A STUDY OF THE INFINITE DIMENSIONAL LINEAR AND SYMPLECTIC GROUPS

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Abstract

By a linear group we shall mean essentially a group of invertible matrices over a ring. Thus, we include in our class of linear groups the 'classical' geometric groups. These are the general linear group, $GL_n(F)$, the orthogonal groups, $O_n(F)$ and the symplectic groups $Sp_n(F)$. The normal and subnormal subgroup structure of these groups is well known and has been the subject of much investigation since the turn of the century. We study here the normal and subnormal structure of some of their infinite dimensional counterparts, namely, the infinite dimensional linear group $GL(\Omega,R)$, for arbitrary rings R, and the infinite dimensional symplectic group $Sp(\Omega,R)$, for commutative rings R with identity. We shall see that a key rôle in the classification of the normal and subnormal subgroups of $GL(\Omega,R)$ and $Sp(\Omega,R)$ is played by the 'elementary' normal subgroups $E(\Omega,R)$ and $ESp(\Omega,R)$. We shall also see that, in the case of the infinite dimensional linear group, the normal subgroup structure depends very much upon the way in which R is generated as a right R-module. We shall also give a presentation for the 'elementary' subgroup $E(\Omega,R)$ when R is a division ring.

Declaration

I declare that the following thesis is a record of research carried out by me, that the thesis is my own composition and that it has not been presented previously in application for a higher degree.

Declaration

I declare that I was admitted in April 1975 under University Court Ordinance General No 12 as a part time research student in the Department of Pure Mathematics.

I certify that David G Arrell has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of doctor of philosophy.

Preface

I should like to express my thanks to my supervisor, Dr E F Robertson, whose help and encouragement I have greatly appreciated.

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CONTENTS

Chapter one

By a linear group we shall mean essentially a group of invertible matrices over a ring. Thus, we include in our class of linear groups the 'classical' geometric groups. These are $GL_n(F)$, the group of bijective linear transformations of an n dimensional vector space V over a field F, the orthogonal groups $O_n(F)$ in V, defined by non-degenerate quadratic forms, and the symplectic group, $Sp_n(F)$ which is defined as the group of isometries of an n dimensional vector space over F equipped with a metric given by a non-degenerate alternate bilinear form.

The study of linear groups dates from the late nineteenth century when Jordan investigated $GL_n(F)$ and $Sp_n(F)$ for $F = Z_p$, p a prime, in his Traité des substitutions (1870). This work was then extended in the twentieth century first by Dickson [12] and then Iwasawa [22] to cover arbitrary base fields. Dieudonné has also contributed substantially to our knowledge of the structure of the classical groups and we summarise some of his work in chapter two.

The object of this thesis is to investigate the structure of certain infinite dimensional linear groups, namely, the infinite dimensional general linear group $GL(\Omega,R)$, the group of units of the endomorphism ring of the free R-module $R^{(\Omega)}$, for an infinite set Ω and an arbitrary ring R, and the infinite dimensional symplectic group $Sp(\Omega,R)$, the group of isometries of an infinite dimensional R-module equipped with an alternate non-degenerate bilinear form.

In chapter three we shall give a classification of the normal and subnormal subgroups of $GL(\Omega,R)$, where R is an arbitrary ring.

If we impose certain finiteness conditions on R, we find that we can improve this classification and details of this improvement are given in chapter six. Chapter four gives a classification of the normal and subnormal subgroups of $Sp(\Omega,R)$. We shall find that certain 'elementary' elements of $GL(\Omega,R)$ play a key rôle in the classification of the normal subgroups of $GL(\Omega,R)$ and in chapter five we give a presentation for the subgroup of $GL(\Omega,R)$ that they generate, when R is a division ring. The work presented here is not the first investigation of the structure of infinite dimensional linear groups. Much work in this area has been carried out by Rosenberg, Spiegel, Robertson and Maxwell and we give a summary of their results in chapter two. We devote the rest of chapter one to listing some definitions and stating some basic results that we shall employ in this thesis.

(a) group theoretic definitions

Definition 1.1. Let G be a group and let $x, y \in G$; by the commutator $[x,y]$ we shall mean the element $x^{-1}y^{-1}xy$ of G. We shall denote $y^{-1}xy$ by x^y.

Definition 1.2. Let X be a subset of a group G; $\langle X \rangle$ shall denote the subgroup of G generated by X, that is, $\langle X \rangle$ is the intersection of all the subgroups of G that contain X. It will also be convenient to write $\langle X \rangle$ as $\langle x : x \in X \rangle$.

Definition 1.3. If X and Y are two subsets of a group G, we shall write:

$[X,Y]$ for $\langle [x,y] ; x, y \in G \rangle$,

X^g for $\langle x^g : x \in X \rangle$ for any $g \in G$,

X^Y for $\langle x^y : x \in X, y \in Y \rangle$.

Lemma 1.1. Let x, y, z be elements of a group G and letH, K, L be subgroups of G. We have

(i) $[xy,z] = [x,z]^y[y,z]$,

(ii) $[x,yz] = [x,z][x,y]^z$,

(iii) if H, K, L are normal subgroups of G then $[HK,L] = [H,L][K,L]$,

(iv) if H, K, L are subgroups of G and if any two of $[H,K,L]$, $[K,L,H]$ and $[L,H,K]$ are contained in a normal subgroup of G then so is the third. (This is often known as Hall's three subgroup lemma.)

Proof. For example [35, pp 42-44].

Definition 1.4. Let G be a group. The subgroup $[G,G]$ is called the derived group of G and is denoted by $\gamma_2(G)$. If $G = \gamma_2(G)$ then we say that G is a perfect group. For any positive integer i define

$$\gamma_{i+1}(G) = [G,\gamma_i(G)]$$

where we take $G = \gamma_1(G)$. The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \geq \gamma_i(G) \geq \dots$$

is called the lower central series of G. If $\gamma_{d+1}(G) = 1$ but $\gamma_d(G) \neq 1$, for some integer d, then we say that G is a nilpotent group of class d. If

$$\bigcap_{i=1}^{\infty} \gamma_i(G) = 1$$

then we say that G is residually nilpotent.

Definition 1.5. Let K be a subgroup of G. The intersection of all those normal subgroups of G that contain K is called the normal closure of K in G. We see that the normal closure of K in G coincides with the subgroup K^G.

Definition 1.7. Let H and K be subgroups of a group G. We denote the subgroup

$$\bigcap_{k \in K} H^k$$

by $\mathrm{Core}_K H$. We see that $\mathrm{Core}_K H$ is the largest subgroup of H that is normalized by K.

Definition 1.8. Let H be a subgroup of a group G. A normal series of length d from H to G is a series

$$H = G_d \leq G_{d-1} \leq \cdots \leq G_1 \leq G_0 = G$$

where G_{i+1} is a normal subgroup of G_i, $i = 0,1,\ldots,d-1$.

Definition 1.9. Let H be a subgroup of a group G. If there exists a normal series from H to G of length d, for some integer $d \geq 1$, then we say that H is a subnormal subgroup of G and we write $H \triangleleft^d G$. The least integer d such that $H \triangleleft^d G$ is called the defect of H in G.

Definition 1.10. Let A be a set and let $\{X_\alpha : \alpha \in A\}$ be a collection of subsets of a group G. We denote the subgroup of G generated by all the X_α by $\langle X_\alpha : \alpha \in A \rangle$.

Definition 1.11. Let A be a set and let $\{X_\alpha : \alpha \in A\}$ be a collection of subsets of a group G. We denote the subset

$$\{x_{\alpha_1} \dots x_{\alpha_k} : k \geq 1,\ \alpha_i \in A,\ x_{\alpha_i} \in X_{\alpha_i},\ i = 1,\dots,k\}$$

by $\prod_{\alpha \in A} X_\alpha$ and call this subset the product or join of the subsets X_α.

Theorem 1.1. If G is a group and $H_1,\dots,H_r$ are subgroups of G such that

$$\gamma_{m_i+1}(H_i^G) \leq H_i, \quad i = 1,\dots r,$$

for some integers $m_i \geq 1$, then $\langle H_1,\dots,H_r\rangle$ is a subnormal subgroup of G with defect at most

$$\sum_{i=1}^{r} m_i + 1.$$

Proof. First notice that if H and K are subgroups of G with $K \leq H$ and $\gamma_{m+1}(H) \leq K$ then $H/\gamma_{m+1}(H)$ is a nilpotent group of class m containing $K/\gamma_{m+1}(H)$ as a subgroup. It follows that K is a subnormal subgroup of H of defect at most m. If we now suppose that H is a normal subgroup of G then it follows that K will be a subnormal subgroup of G of defect at most m+1.

Next notice that if H and K are normal subgroups of G and if M and L are subgroups of G with $L \leq K$, $M \leq H$, $\gamma_{m+1}(K) \leq L$, $\gamma_{n+1}(H) \leq M$ then $\gamma_{n+m+1}(HK) \leq \langle M,L\rangle$ since

$$\gamma_k(HK) \leq \gamma_k(H)\gamma_k(K)\{\prod_{i=1}^{k-1} \gamma_i(H) \cap \gamma_{k-i}(K)\}.$$

If we put $H_0 = \langle H_1,\dots,H_r\rangle$ and $m = \sum_{i=1}^{r} m_i$ we see that $\gamma_{m+1}(H_0^G) \leq H_0$

and the result now follows by our previous remark. (Notice that this result does not agree with that given by Wilson in [40] where he states that

$$\sum_{i=1}^{r} m_i - r + 2$$

is an upper bound for the defect of H_0. To see that Wilson's upper bound cannot be correct, take $m_i = 0$, $i = 1,\ldots,r$.)

(b) ring theoretic definitions

Definition 1.12. An element u of a ring R is said to be a unit of R if u has a multiplicative inverse in R. We shall denote the group of units of R by R*.

Definition 1.13. An element x of R is said to be nilpotent if there exists an integer n such that $x^n = 0$. An ideal p of R is said to be nil if every x in p is nilpotent.

Definition 1.14. Let p, q be two subrings of a ring R. By the product pq we mean the set of all finite sums $\Sigma r_i s_i$, $r_i \in p$, $s_i \in q$. In particular, for any integer $m \geq 2$, we define $R^m = R(R^{m-1})$.

Definition 1.15. A ring R is said to be nilpotent if there exists an integer n such that $R^n = 0$. R is said to be residually nilpotent if

$$\bigcap_{i=1}^{\infty} R^i = 0.$$

Definition 1.16. The union of all the nil ideals of a ring R is called the nil radical of R.

Definition 1.17. A ring R is said to be simple if and only if the only two sided ideals of R are 0 and R.

Definition 1.18. A ring R with identity is said to be local if it has a unique maximal proper ideal. (Equivalently, R is a local ring if the set of non-units of R is an ideal.)

Definition 1.19. We denote the field of p elements, p a prime, by GF(p).

We shall adopt the convention that all R-modules are <u>right</u> R-modules.

Definition 1.20. Let Ω be an infinite set and let R be a ring with identity. Let M be the free R-module $R^{(\Omega)}$ with canonical basis $\{e_\omega : \omega \in \Omega\}$. Let $x \in M$, say

$$x = \Sigma e_\omega x_\omega.$$

We say that x is unimodular if there exist $a_\omega \in R$ such that $\Sigma a_\omega x_\omega = 1$.

Definition 1.21. Let M be an R-module. We shall say that M is Noetherian if every ascending chain of R-submodules terminates after a finite number of steps.

Theorem 1.2. Let M be an R-module. The following conditions are equivalent.

(i) M is Noetherian,

(ii) every submodule of M is finitely generated.

Proof. For example, Cohn [8, pg 35].

Definition 1.22. A ring R is said to be Noetherian if it is Noetherian as an R-module.

Definition 1.23. Let R be a commutative ring with identity and let S denote the set of all non-divisors of zero in R. Let $S^{-1}R$ denote the full ring of fractions of R (e.g. Cohn, [8, pg 394]). We shall call the ideal p of R invertible if there exists an ideal q of $S^{-1}R$ such that $pq = R$. A (commutative) integral domain R is said to be a Dedekind ring if every non-zero ideal is invertible. A local Dedekind ring is called a discrete valuation ring (DVR).

Theorem 1.3. Let R be a DVR and let $m \neq 0$ be the maximal ideal of R. Then

(i) R is Noetherian,

(ii) every ideal of R is principal and is a power of m,

(iii) every non-zero ideal of m contains a power of m,

(iv) m is residually nilpotent.

Proof. For example, Bass [4, pg 137].

(c) general definitions

Definition 1.24. Let Ω be a set and $\leq$ an order on Ω. We shall say that $(\Omega,\leq)$ is an ordered set.

Definition 1.25. Let $(\Omega,\leq)$ be an ordered set. We shall say that Ω is a totally (or linearly) ordered set if and only if $\alpha, \beta \in \Omega$ implies that $\alpha \leq \beta$ or $\beta \leq \alpha$.

Definition 1.26. Let $(\Omega,\leq)$ be an ordered set. We shall say that Ω is well ordered if and only if every non-empty subset of Ω has a least element.

Theorem 1.4. (The well ordering theorem.) If Ω is a set then there is an order $\leq$ on Ω such that $(\Omega,\leq)$ is a well ordered set.

Proof. For example, Halmos [21, pg 68].

In particular we see that any set of cardinal numbers is well ordered. We shall also need

Theorem 1.5. If C is a set of cardinal numbers then there exists a cardinal number greater than each cardinal number in C.

Proof. For example, Stoll [38, pg 120].

Notation: Let $\underline{u}$ be an infinite cardinal number. If α is the ordinal number (i.e. order type) of the set of infinite cardinals less than u, then u is designated by $\aleph_\alpha$.

(d) the infinite dimensional linear group $GL(\Omega,R)$

Let Ω be a set and let R be a ring with identity. By an $\Omega \times \Omega$ matrix A over R we shall mean a mapping

$$A: \Omega \times \Omega \to R.$$
$$(\lambda,\mu) \to A(\lambda,\mu).$$

For fixed $\lambda \in \Omega$ we call $\{A(\lambda,\mu) : \mu \in \Omega\}$ the λth row of A. For fixed $\mu \in \Omega$ we call $\{A(\lambda,\mu) : \lambda \in \Omega\}$ the μth column of A. If, for all $\lambda \in \Omega$, $A(\lambda,\mu) = 0$ for almost all $\mu \in \Omega$ we say that A is row finite. If, for all $\mu \in \Omega$, $A(\lambda,\mu) = 0$ for almost all $\lambda \in \Omega$ then we say that A is column finite. Denote by $Mat(\Omega,R)$ the ring of all $\Omega \times \Omega$ column finite matrices over R. For any $A \in Mat(\Omega,R)$ we shall often abbreviate $A(\lambda,\mu)$ to $A_{\lambda\mu}$ and call $A_{\lambda\mu}$ the (λ,μ)th entry of A.

Let M denote the free R-module $R^{(\Omega)}$ with canonical basis $\{e_\omega : \omega \in \Omega\}$ and let $End_R(M)$ denote the endomorphism ring of M over R. Whenever $X \in End_R(M)$ and $\omega \in \Omega$ then we see that

$$X(e_\omega) = \sum_\alpha e_\alpha x_{\alpha\omega}$$

for some $x_{\alpha\omega} \in R$, $\alpha \in \Omega$, with almost all $x_{\alpha\omega} = 0$. In this way, we see that X defines the $\Omega \times \Omega$ column finite matrix mat(X) over R given by

$$Mat(X)_{\alpha\omega} = x_{\alpha\omega}.$$

Conversely, if $A \in Mat(\Omega,R)$ then we can define the endomorphism end(A) of M by prescribing the effect of end(A) on the canonical basis of M, namely,

$$\mathrm{end}(A)e_\omega = \sum_\alpha e_\alpha A_{\alpha\omega}$$

and then extending by linearity. In this way, we see that there is an isomorphism

$$\Phi:\mathrm{Mat}(\Omega,R) \to \mathrm{End}_R(M)$$

given by $\varphi(A) = \mathrm{end}(A)$ and $\varphi^{-1}(X) = \mathrm{Mat}(X)$. In the sequel we shall often identify A and end(A) and X and Mat(X). We shall refer to $\Omega \times \Omega$ matrices as endomorphisms of M and vice versa. It shall always be understood that we are using the canonical basis of M to make these identifications.

We denote by $GL(\Omega,R)$ the group of units of $\mathrm{End}_R(M)$. Thus we can regard the elements of $GL(\Omega,R)$ as invertible $\Omega \times \Omega$ matrices over R. When Ω is a finite set of cardinality n we shall write $GL(\Omega,R)$ as $GL(n,R)$. Let p be a two-sided ideal of R. We can construct the canonical epimorphism

$$\psi:R \to R/p.$$

If $r \in R$, denote the element $\psi(r) \in R/p$ by $\bar{r}$. We use ψ to construct a canonical epimorphism

$$\psi':GL(\Omega,R) \to GL(\Omega,R/p)$$

by defining $\psi'(A) = \bar{A}$, where $\bar{A}(\lambda,\mu) = \overline{A(\lambda,\mu)}$, for each $A \in GL(\Omega,R)$. Denote the kernel of this epimorphism by $GL(\Omega,p)$ and the inverse image of the centre of $GL(\Omega,R/p)$ by $GL'(\Omega,p)$. Notice that if $A \in GL(\Omega,p)$ then, for all $\lambda, \mu \in \Omega, \lambda \neq \mu$, $A_{\lambda\mu} \equiv 0 \bmod p$ and $A_{\lambda\lambda} \equiv 1 \bmod p$. When Ω has finite cardinality n we write $GL(\Omega,p)$ and $GL'(\Omega,p)$ as $GL(n,p)$ and $GL'(n,p)$ respectively. When R is a commutative ring we have the determinant map $\det:GL(n,R) \to R$. We denote the kernel of this mapping by $SL(n,R)$. We shall call $SL(n,R)$ the special linear group. Let e_{ij}, $1 \leq i,j \leq n$, denote the $n \times n$ matrix with 1 in the (i,j)th position and zeros elsewhere. It is well known that when R is a field, $SL(n,R)$ is generated by all the matrices $1 + re_{ij}$, for all $i,j = 1,\ldots,n$, and all $r \in R$. The matrices $1 + re_{ij}$ are known as transvections or elementary matrices.

When R is a ring without identity, we can embed R in $R^* = Z \times R$ in the usual way (for example, Jacobson, [23, pg 149]). In this way R is a two sided ideal of R^* and so we can think of $GL(\Omega,R)$ as a normal subgroup of $GL(\Omega,R^*)$.

(e) the infinite dimensional symplectic group

Let R be a ring with identity and let Ω be a totally ordered set. Let Ω' be a totally ordered set which is disjoint from and equipotent to Ω with $':\Omega \to \Omega'$ an order preserving bijection. Define $\Omega_1 = \Omega \cup \Omega'$. Let Ω_1 be totally ordered by inheriting the order from Ω and Ω' and with $\omega < \lambda'$, for all $\omega \in \Omega$ and $\lambda' \in \Omega'$. Let M be the free R-module $R^{(\Omega_1)}$. Define the alternate bilinear form $(*,*)$ on M by:

$$\left.\begin{aligned}(e_\omega, e_{\omega'}) &= 1\\ (e_{\omega'}, e_\omega) &= -1\end{aligned}\right\}\quad \text{for all } \omega \in \Omega,$$

$$\left.\begin{aligned}(e_\alpha, e_\beta) &= 0\\ (e_{\alpha'}, e_{\beta'}) &= 0\end{aligned}\right\}\quad \text{for all } \alpha,\beta \in \Omega,$$

$$\left.\begin{aligned}(e_{\alpha'}, e_\beta) &= 0\\ (e_\alpha, e_{\beta'}) &= 0\end{aligned}\right\}\quad \text{for all } \alpha,\beta \in \Omega, \alpha \neq \beta.$$

We define $Sp(\Omega,R)$ to be the group of automorphisms X of M which are such that $(X(x),X(y)) = (x,y)$, for all $x,y \in M$. We see that $Sp(\Omega,R) \leq GL(\Omega_1,R)$. Since Ω_1 is totally ordered, we can think of any $X \in Sp(\Omega,R)$ as an invertible $\Omega_1 \times \Omega_1$ matrix over R partitioned as

$$\begin{array}{cc} \begin{array}{c} \Omega \\ \\ \Omega' \end{array} & \left[\begin{array}{c|c} \quad\quad & \quad\quad \\ \hline \quad\quad & \quad\quad \end{array}\right] \\ & \begin{array}{cc} \Omega & \quad\quad \Omega' \end{array} \end{array}$$

We shall call $X \in Sp(\Omega,R)$ a symplectic transformation (matrix) and we shall call $Sp(\Omega,R)$ the infinite dimensional symplectic group over R.

Let X be a symplectic matrix and suppose that X is partitioned as

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Since

$$\text{(i)} \quad (X(e_\alpha),X(e_\beta)) = 0 \quad \text{for all } \alpha,\beta \in \Omega,$$

$$\text{(ii)} \quad (X(e_{\alpha'}),X(e_{\beta'})) = 0 \quad \text{for all } \alpha',\beta' \in \Omega',$$

$$\text{(iii)} \quad (X(e_\alpha),X(e_{\beta'})) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

$$\text{(iv)} \quad (X(e_{\alpha'}),X(e_\beta)) = \begin{cases} -1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

and since

$$(X(e_\varphi),X(e_\psi)) = \sum_\lambda X_{\lambda\varphi} X_{\lambda'\psi} - \sum_\lambda X_{\lambda'\varphi} X_{\lambda\psi}$$

for $\varphi,\psi \in \Omega_1$ and $\lambda \in \Omega$ we see that

$$(X(e_\varphi),X(e_\psi)) = \sum_\lambda X'_{\varphi\lambda} X_{\lambda'\psi} - \sum_\lambda X'_{\varphi\lambda'} X_{\lambda\psi}$$

Where X' denotes the transpose matrix of X. This means that

$A'D - C'B = I$, $B'C - D'A = -I$, $A'C - C'A = D'B - B'D = 0$. Thus, if we let J denote the $\Omega_1 \times \Omega_1$ matrix

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

we see that X satisfies the matrix equation

$$X'JX = J.$$

This equation shows that X is row finite as well as being column finite. We conclude that whenever $X \in Sp(\Omega,R)$, we can regard X as an invertible row and column finite $\Omega_1 \times \Omega_1$ matrix satisfying the matrix equation $X'JX = J$. Conversely, we may regard this as a defining property of a symplectic matrix; for, if we pick $x,y \in M$ and define

$$(x,y) = \underline{x}'J\underline{y}$$

where $\underline{x}$, $\underline{y}$ are the co-ordinate vectors of x, y with respect to the canonical basis of M, we see that $(*,*)$ is an alternate bilinear form and

$$(X(x),X(y)) = \underline{x}'X'JX\underline{y} = \underline{x}'J\underline{y} = (x,y).$$

Let p be a two sided ideal of R. As in the case of the infinite dimensional general linear group, the canonical epimorphism

$$R \to R/p$$

induces a canonical epimorphism

$$Sp(\Omega,R) \to Sp(\Omega,R/p).$$

We shall denote the kernel of this epimorphism by $Sp(\Omega,p)$ and the inverse image of the centre of $Sp(\Omega,R/p)$ by $Sp'(\Omega,p)$. It is clear that when $X \in Sp(\Omega,p)$, $X_{\alpha\beta} \equiv 0 \bmod p$ and $X_{\alpha\alpha} \equiv 1 \bmod p$, for all $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$.

When Ω is a set of finite cardinality n, we denote $Sp(\Omega,R)$, $Sp(\Omega,p)$ and $Sp'(\Omega,p)$ by $Sp(2n,R)$, $Sp(2n,p)$ and $Sp'(2n,p)$ respectively.

Chapter two

The study of the classical groups has occupied the attention of many group theorists for several years. We summarize here some of the theorems concerning the classification of the normal and subnormal subgroups of the classical groups that have relevance to the main results of the thesis.

1. The classification of the normal subgroups of $GL(\Omega,R)$.

We begin with the finite dimensional linear group $GL(n,R)$, $|\Omega| = n$. Dieudonné [15] and Artin [2] considered $GL(n,R)$ when R is a field. They showed that, unless $n = 2$ ~~or~~ and R is GF(2) or GF(3), the normal subgroups of $GL(n,R)$ either lie in the centre of $GL(n,R)$ or contain the special linear group $SL(n,R)$. If $n \geq 3$ and R is either a commutative local ring or the ring of rational integers then the results of Klingenberg [26] and Mennicke [29] show that any normal subgroup H of $GL(n,R)$ is sandwiched between $GL'(n,p)$ and $SL(n,R) \cap GL(n,p)$, that is

$$SL(n,R) \cap GL(n,p) \leq H \leq GL'(n,p)$$

for some unique two sided ideal p of R. However, this result breaks down for arbitrary rings R, even for large n.

Bass has obtained results of this nature by the introduction of stable range conditions. Let R be a ring with identity. We say that R satisfies the condition $SR_n(R)$ if whenever $m \geq n$ and $\alpha = (a_1,\ldots,a_m) \in R^{(m)}$ is unimodular then there exist $a'_i = a_i + b_i a_m$ with $b_i \in R$, $1 \leq i < m$, such that $(a'_1,\ldots,a'_{m-1}) \in R^{(m-1)}$ is unimodular.

Let $E(n,R)$ be the subgroup of $GL(n,R)$ generated by all the transvections $1 + re_{ij}$. Let $E(n,p)$ denote the normal closure of $\{1 + re_{ij} : r \in p,\ 1 \leq i,j \leq n,\ i \neq j\}$ in $E(n,R)$, for any two sided ideal p of R. Bass proves in [4]

Theorem 2.1. Let R be a ring with identity and suppose that

(i) R satisfies condition $SR_n(R)$ and

(ii) $n \geq 3$.

A subgroup H of $GL(n,R)$ is normalized by $E(n,R)$ if and only if there is a two sided ideal q of R such that

$$E(n,q) \leq H \leq GL'(n,q).$$

Other similar results are obtained in [24] and [25].

Wilson has obtained a convenient sandwiching of not only normal subgroups of $GL(n,R)$ but also subnormal subgroups of $GL(n,R)$, when R is a commutative ring with identity. In [40] he proves

Theorem 2.2. Let R be a commutative ring with identity, $n \geq 3$ and $E(n,R) \leq G \leq GL(n,R)$ and suppose that $H \triangleleft^d G$. Then

$$E(n,p^{f(d,n)}) \leq H \leq GL'(n,p)$$

for some ideal p of R, where

$$f(d,3) = (88(28^{d-1}) - 7)/27 \quad \text{and}$$

$$f(d,n) = (7^d - 1)/6, \quad \text{for } n \geq 4.$$

A special case of this result is

Theorem 2.3. Let R be a commutative ring with identity and let $n \geq 4$. If H is a normal subgroup of GL(n,R) then

$$E(n,p) \leq H \leq GL'(n,p)$$

for some uniquely determined ideal p of R.

It was announced by Golubcik in [18] that the conclusion of Theorem 2.3 holds for any integer $n \geq 3$, but no proof of this assertion has appeared. If we now let $C(n,p) = [GL(n,R),GL'(n,p)]$ then a corollary of Theorem 2.2 is

Theorem 2.4. Let R be either a commutative local ring or the ring of rational integers and let $n \geq 3$. A subgroup H of GL(n,R) is subnormal if and only if

$$C(n,p^m) \leq H \leq GL'(n,p)$$

for some ideal p of R and some integer m. Moreover, the least such m and the defect d of H in GL(n,R) are related as follows:

$$d - 1 \leq m \leq f(d,n)$$

where f is as defined in Theorem 2.2.

In fact, Wilson gives Theorem 2.4 in a more general form in [40]. He says that a commutative ring R satisfies condition R_n if and only if, for every normal subgroup H of GL(n,R)

$$C(n,p) \leq H \leq GL'(n,p)$$

for some ideal p of R. It follows from the results of Klingenberg [26] and Mennicke [29] that if R is a commutative local ring or the ring of rational integers then R satisfies condition R_n. Wilson's version of Theorem 2.4 is the more general one which requires R to satisfy condition R_n. He also shows in [40] that R satifies condition R_n if and only if

$$C(n,p) = E(n,p)^{GL(n,R)}$$

for all ideals p of R.

These results give a very satisfactory classification of subnormal subgroups of $GL(n,R)$: the subnormal subgroups are determined by the ideals p of R and the subgroups of the nilpotent groups $GL'(n,p)/C(n,p^m)$. We shall see in chapter three that it is possible to give a similar classification of the subnormal subgroups of the infinite dimensional general linear group.

We now consider the normal subgroup structure of $GL(\Omega,R)$, when Ω is an infinite set. Let $EF(\Omega,R)$ be the subgroup of $GL(\Omega,R)$ generated by all the matrices of the form $1 + re_{\lambda\mu}$ with $r \in R$, $\lambda,\mu \in \Omega$, $\lambda \neq \mu$, where $e_{\lambda\mu}$ denotes the $\Omega \times \Omega$ matrix with 1 in the (λ,μ)th position and zeros elsewhere. Let $EF(\Omega,p)$ denote the normal closure of $\{1 + re_{\lambda\mu} : \lambda,\mu \in \Omega, \lambda \neq \mu, r \in p\}$, for p a right ideal of R, in $EF(\Omega,R)$. When $\Omega = N$, the group $EF(N,R)$ is just the stable linear group of Bass [3]. The work of Bass [3] and Robertson [31] combine to give

Theorem 2.5. Let R be a ring with identity and let Ω be an infinite set; let H be a subgroup of $GL(\Omega,R)$. If H is normalized by $EF(\Omega,R)$ then

$$EF(\Omega,p) \leq H \leq GL'(\Omega,p)$$

for some uniquely determined two sided ideal p of R.

We see that 'sandwich' type results for normal subgroups of $GL(\Omega,R)$ are possible with no restriction on R when Ω is an infinite set.

We shall see that the 'sandwiches' of Theorem 2.5 are very 'thick' in the sense that many normal subgroups lie between $EF(\Omega,p)$ and $GL'(\Omega,p)$. We shall show that it is possible to make these sandwiches thinner but only at the cost of introducing some conditions on the ring R, (contrary to the results of Maxwell [27]).

Finally, we remark that a complete classification of the normal subgroups of $GL(\Omega,R)$, when Ω is an infinite set and R is a division ring, was given by Rosenberg in [36]. We give a brief outline of that classification.

Let $\underline{a}$ be an infinite cardinal and let $G_{\underline{a}(1)}$ denote the subgroup of $GL(\Omega,R)$ generated by all $X \in GL(\Omega,R)$ with ranges of dimension less than $\underline{a}$. Rosenberg proves

Theorem 2.6. Let R be a division ring, let Ω be an infinite set and let H be a proper normal subgroup of $GL(\Omega,R)$. Then either

(i) $H = K \times G_{\underline{a}(1)}$, where K is a multiplicative subgroup of Z^*, the centre of R^* and $\underline{a}$ is an infinite cardinal such that $\aleph_0 < \underline{a} \leq |\Omega|$, or

(ii) $H \leq Z^* \times G_{\aleph_0(1)}$.

We shall attempt, in chapter six, to give an analogous classification of the normal subgroups of $GL(\Omega,R)$ for rings R which are not necessarily division rings.

2. the classification of the normal subgroups of $Sp(\Omega,R)$

Let $(*,*)$ be the non-degenerate alternate bilinear form associated with $Sp(\Omega,R)$ and let M be the free R-module with symplectic basis $\{e_\lambda, e_{\lambda_1} : \lambda \in \Omega\}$. For any unimodular $x \in M$ and any $a \in R$ define the transvection $t(a,x)$ by

$$t(a,x)m = m + xa(m,x)$$

whenever $m \in M$. Let $Ep(\Omega,R)$ be the subgroup of $Sp(\Omega,R)$ generated by all the transvections $t(a,x)$. Dieudonné in [13] and [14] has studied $Sp(\Omega,R)$ when Ω is finite, $|\Omega| = n$, and when R is a division ring. He proves

Theorem 2.7. When R is a division ring of more than 25 elements $Ep(2n,R)/Z(Ep(2n,R))$ is simple.

The infinite dimensional symplectic group is treated in [37] when R is a division ring. There it is shown that $Ep(\Omega,R)$ has trivial centre and

Theorem 2.8. When R is a division ring and Ω is an infinite set, $Ep(\Omega,R)$ is simple.

Maxwell in [28] studies $Sp(\Omega,R)$ when Ω is an infinite set and R is a commutative ring with identity. We now meet again the familiar sandwich results of §1. Maxwell proves

Theorem 2.9. Let R be a commutative ring with identity and suppose that 2 is a unit of R. The following assertions are equivalent.

(i) H is a subgroup of $Sp(\Omega,R)$ normalized by $Ep(\Omega,R)$

(ii) there exists a unique ideal p of R such that

$$Ep(\Omega,p) \leq H \leq Sp'(\Omega,p).$$

We shall study $Sp(\Omega,R)$ when Ω is infinite and R is a ring with identity and obtain a classification of the subnormal subgroups of $Sp(\Omega,R)$ when R is commutative. This classification will be similar to that of Theorem 2.2.

3. presentations of general and stable linear groups

The stable group $EF(N,R)$ has received much attention not only in the classification of its normal subgroups but also because of its relevance to algebraic K-theory. We have seen that $EF(N,R)$ is generated by the elementary matrices $1 + re_{ij}$. Call these elementary matrices $t_{ij}(r)$. For each integer i, j and $r \in R$ introduce the symbols $s_{ij}(r)$. These symbols we call Steinberg symbols. We define the group $StF(N,R)$ to be the group generated by all the Steinberg symbols $s_{ij}(r)$ subject to the relations

$$\text{(i)} \quad s_{ij}(r)s_{ij}(s) = s_{ij}(r+s)$$

$$\text{(ii)} \quad [s_{ij}(r),s_{k\ell}(s)] = \begin{cases} 1 & j \neq k,\ i \neq \ell, \\ s_{i\ell}(rs) & j = k,\ i \neq \ell, \\ s_{kj}(-sr) & j \neq k,\ i = \ell. \end{cases}$$

We see that $EF(N,R)$ is isomorphic to a homomorphic image of $StF(N,R)$

because of the epimorphism

$$\varphi: s_{ij}(r) \to t_{ij}(r)$$

$K_2(R)$ is just the kernel of φ. Finding a presentation for $K_2(R)$ is a long standing problem and the solution is still unknown, even when R is a division ring - despite a paper of Green [19]. However, when R is a division ring, a presentation for EF(N,R) has been given. The presentation we now outline is due to Green [20] and is based on detailed computations in StF(N,R) due to Steinberg which can be found in Milnor [30].

Let R be a division ring. If $x \in R^*$, make the following definitions:

$$m_{ij}(x) = t_{ij}(x)t_{ji}(-x^{-1})t_{ij}(x)$$

$$d_{ij}(x) = m_{ij}(x)m_{ij}(-1).$$

Notice that $d_{ij}(x)$ is the diagonal matrix with x in the (i,i)th place and x^{-1} in the (j,j)th place and ones elsewhere on the diagonal. If $x,y \in R^*$, define $a_1(x,y)$ to be the matrix $d_{12}(x)d_{12}(y)d_{12}(yx)^{-1}$, that is, the diagonal matrix with $[x^{-1},y^{-1}]$ in position (1,1) and ones else where on the diagonal. The main theorem of [20] is

Theorem 2.10. If $n \geq 3$ then E(n,R) is generated by the symbols $t_{ij}(x)$ where $i \neq j$, $1 \leq i,j \leq n$ and $x \in R$ subject to the relations

(A) $\quad t_{ij}(u)t_{ij}(v) = t_{ij}(u+v) \qquad u,v \in R$

(B) $$[t_{ij}(u),t_{k\ell}(v)] = \begin{cases} 1 & \text{if } j \neq k,\ i \neq \ell \\ t_{i\ell}(uv) & \text{if } j = k,\ i \neq \ell \\ t_{kj}(-vu) & \text{if } j \neq k,\ i = \ell \end{cases}$$

(C) let $u,v \in R^*$, if $j \neq 1$ then

$$a_1(u,v) = d_{1j}(u)d_{1j}(v)d_{1j}(vu)^{-1}$$

if 1, i, j are all distinct then

$$a_1(u,v) = d_{ij}(u)d_{ij}(v)d_{ij}(vu)^{-1}d_{1i}([u,v]).$$

(D) let $u_j, v_j \in R^*$,

$$\prod_{j=1}^{s} a_1(u_j,v_j)^{\varepsilon_j} = 1 \text{ if } \prod_{j=1}^{s} [u_j,v_j]^{\varepsilon_j} = 1,$$

$\varepsilon_j = \pm 1$.

The methods of [20] also prove that, when R is a division ring, EF(N,R) is generated by the symbols $t_{ij}(x)$, $i \neq j$, $i,j \in N$, $x \in R$ subject to the relations (A), (B), (C) and (D). We shall give similar presentations for the groups analogous to EF(N,R) that are introduced in chapter three.

Chapter three

The aim of this chapter is to give a classification of the normal and subnormal subgroups of the infinite dimensional linear group $GL(\Omega,R)$, the group of column finite invertible $\Omega \times \Omega$ matrices over R. The form of this classification will be similar to that given for the normal and subnormal subgroups of $GL(n,R)$ by Wilson in Theorem 2.2. The methods we use are similar to those of [31] and [40]. Part of the work contained in this chapter is contained in a joint paper of mine with Robertson [1].

Unless otherwise stated, throughout this chapter R shall denote a ring with identity and Ω shall be an infinite set.

Definition 3.1. Let $X \in GL(\Omega,R)$. We shall denote by $J(X)$ the two sided ideal of R generated by the matrix entries $X_{\alpha\beta}$ and $X_{\alpha\alpha} - X_{\beta\beta}$, for all $\alpha,\beta \in \Omega$, $\alpha \neq \beta$. We shall call $J(X)$ the level of X (c.f. [40]).

Definition 3.2. Let $H \leq GL(\Omega,R)$. We shall denote by $J(H)$ the ideal

$$\sum_{X \in H} J(X).$$

We shall call $J(H)$ the level of H.

Lemma 3.1. (i) If R is a commutative ring then for all $X, Y \in GL(\Omega,R)$, $J(X^Y) = J(X)$.

(ii) If $X, Y \in GL(\Omega,R)$ and $(X-I)$ has less than card Ω non-zero rows then $J(X^Y) = J(X)$.

(iii) If R is a commutative ring, if $H \leq GL(\Omega,R)$ and if

$Y \in GL(\Omega, R)$ then $J(H^Y) = J(H)$.

(iv) If $H \leq GL(\Omega, R)$, if $Y \in GL(\Omega, R)$ and if, for every $X \in H$, $(X-I)$ has less than card Ω non-zero rows, then $J(H^Y) = J(H)$.

Proof. (i) Let $X, Y \in GL(\Omega, R)$, $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. Then

$$(X^Y)_{\alpha\beta} = \sum_{\substack{\gamma,\delta \\ \gamma\neq\delta}} Y^{-1}_{\alpha\gamma} X_{\gamma\delta} Y_{\delta\beta} + \sum_{\gamma} Y^{-1}_{\alpha\gamma} X_{\gamma\gamma} Y_{\gamma\beta}$$

$$\equiv \sum_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\beta} X_{\delta\delta} \qquad \text{mod } J(X), \text{ for some } \delta \in \Omega$$

since R is commutative. But $\sum\limits_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\beta} = 0$ so that $(X^Y)_{\alpha\beta} \equiv 0 \text{ mod } J(X)$. Also

$$(X^Y)_{\alpha\alpha} - (X^Y)_{\beta\beta} = \sum_{\substack{\gamma,\delta \\ \gamma\neq\delta}} Y^{-1}_{\alpha\gamma} X_{\gamma\delta} Y_{\delta\alpha} - \sum_{\substack{\gamma,\delta \\ \gamma\neq\delta}} Y^{-1}_{\beta\gamma} X_{\gamma\delta} Y_{\delta\beta} + \sum_{\gamma} Y^{-1}_{\alpha\gamma} X_{\gamma\gamma} Y_{\gamma\alpha} - \sum_{\gamma} Y^{-1}_{\beta\gamma} X_{\gamma\gamma} Y_{\gamma\beta}$$

$$\equiv \sum_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\alpha} X_{\alpha\alpha} - \sum_{\gamma} Y^{-1}_{\beta\gamma} Y_{\gamma\beta} X_{\beta\beta} \qquad \text{mod } J(X)$$

since R is commutative. But $\sum\limits_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\alpha} = \sum\limits_{\gamma} Y^{-1}_{\beta\gamma} Y_{\gamma\beta} = 1$ so that $(X^Y)_{\alpha\alpha} - (X^Y)_{\beta\beta} \equiv 0 \text{ mod } J(X)$. We conclude that $J(X^Y) \leq J(X)$. But the choice of X and Y was arbitrary so that $J(X) = J(X^{YY^{-1}}) \leq J(X^Y)$ and this establishes (i).

(ii) We are now not able to assume that R is commutative but by hypothesis there exists $\varphi \in \Omega$ such that $X_{\varphi\varphi} = 1$. Since $X_{\alpha\alpha} - X_{\varphi\varphi} \in J(X)$, for all $\alpha \in \Omega$, $\alpha \neq \varphi$, we see that $X_{\alpha\alpha} - 1 \in J(X)$, for all $\alpha \in \Omega$, $\alpha \neq \varphi$. Clearly $X_{\varphi\varphi} - 1 \in J(X)$ and we deduce that for all $\alpha \in \Omega$, $X_{\alpha\alpha} - 1 \in J(X)$. So we have that

$$(X^Y)_{\alpha\beta} \equiv \sum_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\beta} \equiv 0 \qquad \text{mod } J(X)$$

and that

$$(X^Y)_{\alpha\alpha} - (X^Y)_{\beta\beta} \equiv \sum_{\gamma} Y^{-1}_{\alpha\gamma} Y_{\gamma\alpha} - \sum_{\gamma} Y^{-1}_{\beta\gamma} Y_{\gamma\beta} \quad \mathrm{mod}\ J(X)$$

$$\equiv 0 \quad \mathrm{mod}\ J(X).$$

The proof of (ii) now follows as in (i), since Y is column finite.

(iii) Since R is commutative we see that

$$J(H^Y) = \sum_{A \in H^Y} J(A) = \sum_{X \in H} J(X^Y) = \sum_{X \in H} J(X) = J(H).$$

(iv) The equalities above hold in this case also, since for all $X \in H$, $(X-I)$ has less than card Ω non-zero rows and we can apply (ii).

Definition 3.3. Let $\Lambda \subset \Omega$, $\mu \in \Omega-\Lambda$, p be a right ideal of R and $f:\Lambda \to p$. We shall adopt the convention that f extends to a map $f:\Omega \to p$ by defining $f(\omega) = 0$ for all $\omega \in \Omega-\Lambda$. We define the automorphism $t(\Lambda,f,\mu)$ by

$$t(\Lambda,f,\mu)e_\rho = e_\rho + e_\mu f(\rho), \text{ for all } \rho \in \Omega,$$

where $\{e_\rho : \rho \in \Omega\}$ is the canonical basis of the free R-module $R^{(\Omega)}$ associated with $GL(\Omega,R)$. It is clear that $t(\Lambda,f,\mu) \in GL(\Omega,R)$. Indeed, we can think of $t(\Lambda,f,\mu)$ as an $\Omega \times \Omega$ matrix that differs from the identity matrix I only in the μth row and the non-zero entries of the μth row of $t(\Lambda,f,\mu) - I$ are indexed by Λ. For this reason we call the $t(\Lambda,f,\mu)$ elementary matrices. Clearly $t(\Lambda,f,\mu)^{-1} = t(\Lambda,-f,\mu)$ and $t(\Lambda,f,\mu)t(\Lambda,g,\mu) = t(\Lambda,f+g,\mu)$. The next lemma gives two commutator relations involving elementary matrices and the relations will be used frequently in the rest of this chapter.

Lemma 3.2. Let $t_1 = t(\Lambda_1, f, \mu)$ and $t_2 = t(\Lambda_2, g, \rho)$ be two elementary matrices. Then

(i) if $\mu \notin \Lambda_2$ then $[t_1, t_2] = t(\Lambda_2, f(\rho)g, \mu)$,

(ii) if $\rho \notin \Lambda_1$ then $[t_1, t_2] = t(\Lambda_1, -g(\mu)f, \rho)$.

Proof. We shall prove this lemma by examining the effect of the commutator $[t_1, t_2]$ on the canonical basis $\{e_\alpha : \alpha \in \Omega\}$.

(i) Suppose $\mu \notin \Lambda_2$. Then if $\alpha \notin \Lambda_2$

$$[t_1, t_2]e_\alpha = t_1^{-1}t_2^{-1}(e_\alpha + e_\mu f(\alpha)) = e_\alpha$$

while if $\alpha \in \Lambda_2$

$$\begin{aligned}[t_1, t_2]e_\alpha &= t_1^{-1}t_2^{-1}(e_\alpha + e_\mu(f(\alpha) + f(\rho)g(\alpha))) \\ &= e_\alpha + e_\mu f(\rho)g(\alpha).\end{aligned}$$

Thus $[t_1, t_2] = t(\Lambda_2, f(\rho)g, \mu)$. (ii) is proved in a similar way.

Definition 3.4. $E(\Omega, R)$ is defined to be the subgroup of $GL(\Omega, R)$ generated by $\{t(\Lambda, f, \mu) : \Lambda \subset \Omega,\ \mu \in \Omega - \Lambda,\ f : \Lambda \to R\}$. We see that if $X \in E(\Omega, R)$ then X differs from I in at most finitely many rows.

Definition 3.5. For any right ideal p of R, $E(\Omega, p)$ is defined to be the normal closure of $\{t(\Lambda, f, \mu) : \Lambda \subset \Omega,\ \mu \in \Omega - \Lambda,\ f : \Lambda \to p\}$ in $E(\Omega, R)$.

If $\Lambda = \{\lambda\}$ and $f : \Lambda \to R$ with $f(\lambda) = x$, we shall write $t(\lambda, x, \mu)$ in place of $t(\Lambda, f, \mu)$.

Definition 3.6. $EF(\Omega, R)$ is defined to be the subgroup of $E(\Omega, R)$ generated by $\{t(\lambda, x, \mu) : \lambda, \mu \in \Omega,\ \lambda \neq \mu,\ x \in R\}$.

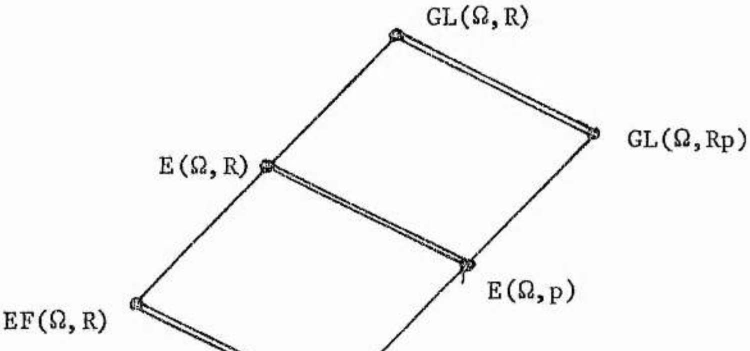
Definition 3.7. For any right ideal p of R, $EF(\Omega,p)$ is defined to be the normal closure of $\{t(\lambda,x,\mu) : \lambda,\mu \in \Omega,\ \lambda \neq \mu,\ x \in p\}$ in $EF(\Omega,R)$.

We have now defined the 'elementary' subgroups $E(\Omega,R)$, $E(\Omega,p)$, $EF(\Omega,R)$ and $EF(\Omega,p)$. It is these subgroups that will play a key rôle in the classification of the normal and subnormal subgroups of $GL(\Omega,R)$. Because of this, we shall now establish some elementary properties of these subgroups.

Lemma 3.4. $E(\Omega,p) \leq GL(\Omega,Rp)$, for any right ideal p of R.

Proof. It is clear that if $t(\Lambda,f,\mu)$ is an elementary matrix generating $E(\Omega,p)$ then $t(\Lambda,f,\mu) \in GL(\Omega,Rp)$ since $f(\lambda) \in p$ for all $\lambda \in \Lambda$. It follows that $E(\Omega,p) \leq GL(\Omega,Rp)$ since $GL(\Omega,Rp)$ is a normal subgroup of $GL(\Omega,R)$.

We now see that the subgroups introduced in definitions 3.4 - 3.7 satisfy the relations implied in the following diagram:



(A single line denotes subgroup inclusion, a double line denotes normal subgroup inclusion.)

We shall now show that $E(\Omega,R)$ and $E(\Omega,p)$ are also normal subgroups of $GL(\Omega,R)$. We begin with

Lemma 3.5. If p is a two sided ideal of R, if $X \in GL(\Omega,p)$ and if $\lambda \in \Omega$ then there exists $Y \in E(\Omega,p)$ such that $Y(e_\lambda) = X(e_\lambda)$.

Proof. The proof of this lemma and the proofs of lemmas 3.6, 3.7 and 3.8 are based on Proposition 2.1 of [27]: however, although that proposition required that R be commutative, our lemma does not. Let $X(e_\lambda) = \Sigma e_\alpha X_{\alpha\lambda}$. Since X is column finite, there exists $\beta \in \Omega$, $\beta \neq \lambda$ such that $X_{\beta\lambda} = 0$. Define $t_1 = t(\beta,-1,\lambda)$ and for each $\alpha \in \Omega$, $\alpha \neq \lambda$, define

$$t_{2\alpha} = t(\lambda, X_{\alpha\lambda}, \alpha) \text{ and}$$
$$t_{2\lambda} = t(\lambda, X_{\lambda\lambda} - 1, \beta).$$

Since X is column finite, $t_{2\alpha} \neq I$ for only finitely many α so let

$$t_2 = \prod_{t_{2\alpha} \neq I} t_{2\alpha}.$$

Next define $t_3 = t(\Omega-\{\beta\}, f, \beta)$ where $f(\alpha) = -(X_{\lambda\lambda} - 1)X^{-1}_{\lambda\alpha}$ for $\alpha \in \Omega-\{\beta\}$. Then t_2, $t_3 \in E(\Omega,p)$ and so $t_3(t_2^{\,t_1}) \in E(\Omega,p)$, since $E(\Omega,p)$ is a normal subgroup of $E(\Omega,R)$. But $t_3(t_2^{\,t_1})e_\lambda = X(e_\lambda)$ and so we can take $Y = t_3(t_2^{\,t_1})$.

Lemma 3.6. For any right ideal p of R, $E(\Omega,p)$ is a normal subgroup of $GL(\Omega,R)$.

Proof. It will be sufficient to show that if $X \in GL(\Omega,R)$ and $t(\Lambda,f,\mu)$ is a generator of $E(\Omega,p)$ then $t^X \in E(\Omega,p)$. For, if $Y \in E(\Omega,p)$ then $Y = t_1^{X_1} \ldots t_r^{X_r}$, where the $X_i \in E(\Omega,R)$, $i = 1,\ldots,r$ and the $t_1,\ldots,t_r$ are elementary matrices from thegenerating set of $E(\Omega,p)$; then $Y^X = t_1^{X_1 X} \ldots t_r^{X_r X}$ and hence $Y^X \in E(\Omega,p)$.

By Lemma 3.5, there exists $Y \in E(\Omega,R)$ such that $Y(e_\mu) = X^{-1}(e_\mu)$. By examining the effect of t^{XY} on the canonical basis of $R^{(\Omega)}$, we see that $t^{XY} = t(\Lambda',g,\mu)$, for some $\Lambda' \subset \Omega$ and $g:\Lambda' \to p$ and deduce that $t^{XY} \in E(\Omega,p)$. Hence $t^X \in E(\Omega,p)$, since $E(\Omega,p)$ is a normal subgroup of $E(\Omega,R)$.

Lemma 3.7. For any right ideal p of R, $E(\Omega,p) = [E(\Omega,R),E(\Omega,p)]$.

Proof. We see, by the definition of $E(\Omega,p)$, that $[E(\Omega,R),E(\Omega,p)] \leq E(\Omega,p)$. We must show the opposite inclusion. It will suffice to show that if $t(\Lambda,f,\mu)$ is a generator of $E(\Omega,p)$ then $t(\Lambda,f,\mu) \in [E(\Omega,R),E(\Omega,p)]$. Suppose first that there exists $\alpha \in \Omega-\Lambda$ such that $\alpha \neq \mu$. Then, by Lemma 3.2, $t(\Lambda,f,\mu) = [t(\Lambda,f,\alpha),t(\alpha,-1,\mu)]$. If $\Omega = \Lambda \cup \{\mu\}$ then write Λ as the disjoint union of the non-empty subsets Λ_1 and Λ_2 of Λ. Then, from the argument above we see that $[E(\Omega,R),E(\Omega,p)]$ contains each of $t(\Lambda_1,f|_{\Lambda_1},\mu)$ and $t(\Lambda_2,f|_{\Lambda_2},\mu)$ and hence contains their product which is $t(\Lambda,f,\mu)$.

We now see immediately that

Corollary 3.1. $E(\Omega,R)$ is a perfect group.

Whenever p is a right ideal of R, we may ask when is $E(\Omega,p)$ perfect? We shall investigate this question after we have proved our classification theorem for the normal and subnormal sugroups of $GL(\Omega,R)$.

For the moment we continue by giving some more elementary properties of $E(\Omega,R)$.

Lemma 3.8. For any two sided ideal p of R, $E(\Omega,p) = [E(\Omega,R),GL'(\Omega,p)]$.

Proof. Since $E(\Omega,p) \leq GL(\Omega,p) \leq GL'(\Omega,p)$, we see that $E(\Omega,p) = [E(\Omega,R),E(\Omega,p)] \leq [E(\Omega,R),GL'(\Omega,p)]$. It remains to prove the opposite inclusion. We first show that $E(\Omega,p) = [E(\Omega,R),GL(\Omega,p)]$. It will be sufficient to show that if $t = t(\Lambda,f,\mu)$ is a generator of $E(\Omega,R)$ and $X \in GL(\Omega,p)$ then $[t,X] \in E(\Omega,p)$ since $E(\Omega,p)$ is a normal subgroup of $GL(\Omega,R)$. By Lemma 3.5 there exists $Y \in E(\Omega,p)$ such that $Y(e_\mu) = X^{-1}(e_\mu)$. Let $Z = XY$. By examining the effect of $[t,Z]$ on the canonical basis of $R^{(\Omega)}$ we see that $[t,Z] \in E(\Omega,p)$. But

$$[t,Z] = [t,XY] = [t,Y][t,X]^Y$$

and so $[t,X] = ([t,Y]^{-1}[t,Z])^{Y^{-1}} \in E(\Omega,p)$ by the definition of $E(\Omega,p)$ and by Lemma 3.6. However, $[E(\Omega,R),GL'(\Omega,p)] \leq GL(\Omega,p)$, by the definition of $GL'(\Omega,p)$. Hence $[E(\Omega,R),GL'(\Omega,p),E(\Omega,R)] \leq E(\Omega,p)$ and so $[E(\Omega,R),E(\Omega,R),GL'(\Omega,p)] \leq E(\Omega,p)$ by Hall's three subgroup lemma and we conclude that $[E(\Omega,R),GL'(\Omega,p)] \leq E(\Omega,p)$ in view of Corollary 3.1.

Lemma 3.9. For any right ideals p, q of R, $E(\Omega,p+q) = E(\Omega,p)E(\Omega,q)$.

Proof. It is clear that $E(\Omega,p) \leq E(\Omega,p+q)$ amd that $E(\Omega,q) \leq E(\Omega,p+q)$. Hence $E(\Omega,p)E(\Omega,q) \leq E(\Omega,p+q)$. To show the opposite inclusion it will be sufficient to show that if $t = t(\Lambda,f,\mu)$ is any generator of $E(\Omega,p+q)$ then $t \in E(\Omega,p)E(\Omega,q)$. Let $\lambda \in \Lambda$, then $f(\lambda) \in p+q$, say $f(\lambda) = r_\lambda + s_\lambda$ for $r_\lambda \in p$ and $s_\lambda \in q$. Define $g:\Lambda \to p$ and $h:\Lambda \to q$ by $g(\lambda) = r_\lambda$ and

$h(\lambda) = s_\lambda$. Then $t = t(\Lambda, g, \mu) t(\Lambda, h, \mu)$ and hence $t \in E(\Omega, p) E(\Omega, q)$.

The classification of normal and subnormal subgroups involves the inverse image of the centre of $GL(\Omega, R/p)$, for various ideals p of R. Our next results give some information about central elements of $GL(\Omega, R)$.

Lemma 3.10. If $X \in GL(\Omega, R)$ commutes with $t(\lambda, x, \rho)$

(i) for all $\lambda \neq \rho$ then $xX_{\lambda\beta} = 0$ for all $\beta \neq \lambda$ and $xX_{\lambda\lambda} = X_{\rho\rho}x$,

(ii) for all $\rho \neq \lambda$ then $X_{\alpha\rho}x = 0$ for all $\alpha \neq \rho$ and $X_{\rho\rho}x = xX_{\lambda\lambda}$.

Proof. Let $E = t(\lambda, x, \rho) - I$. Then $X \in GL(\Omega, R)$ commutes with $t(\lambda, x, \rho)$ if and only if X commutes with E. Now

$$(XE)_{\alpha\beta} = \begin{cases} 0 & \beta \neq \lambda \\ X_{\alpha\rho}x & \beta = \lambda \end{cases}$$

and

$$(EX)_{\alpha\beta} = \begin{cases} 0 & \alpha \neq \rho \\ xX_{\lambda\beta} & \alpha = \rho. \end{cases}$$

Thus if X commutes with $t(\lambda, x, \rho)$ we have that $(XE)_{\alpha\beta} = (EX)_{\alpha\beta}$, for all $\alpha, \beta \in \Omega$. Hence

$$\begin{aligned} xX_{\lambda\beta} &= 0 \quad \text{whenever } \beta \neq \lambda, \\ X_{\alpha\rho}x &= 0 \quad \text{whenever } \alpha \neq \rho \text{ and} \\ xX_{\lambda\lambda} &= X_{\rho\rho}x. \end{aligned}$$

The result now follows.

Corollary 3.2. The centre of $GL(\Omega, R)$ comprises rI, for all central units r of R. The centre of $E(\Omega, R)$ is trivial.

Proof. If $X \in Z(GL(\Omega,R))$ then X commutes with $t(\lambda,1,\mu)$ for all $\lambda,\mu \in \Omega$, $\lambda \neq \mu$. By Lemma 3.10 we see that all the off diagonal entries of X are zero and that $X_{\lambda\lambda} = X_{\mu\mu}$ for all $\lambda,\mu \in \Omega$, $\lambda \neq \mu$. Call this common diagonal entry r, so that $X = rI$ and we see that r is a unit in R since X is an invertible matrix. Also, since X commutes with $t(\lambda,x,\mu)$ for all $x \in R$ it follows that r is central. It is clear that all matrices of the form rI, where r is a central unit of R, are in the centre of $GL(\Omega,R)$: we deduce that $Z(GL(\Omega,R))$ is just the set $\{rI : r \text{ is a central unit of } R\}$. Finally, we remark that elements of $E(\Omega,R)$ differ from the identity matrix in at most finitely many rows and deduce that $E(\Omega,R)$ has trivial centre.

Lemma 3.11. Let $X \in GL(\Omega,R)$. If, for some two sided ideal p of R, $X \in GL'(\Omega,p)$ then $J(X) \leq p$.

Proof. If $X \in GL'(\Omega,p)$ then let $\bar{\ }$ denote images under the group homomorphism $GL(\Omega,R) \to GL(\Omega,R/p)$ induced by the projection $R \to R/p$. Hence $\overline{X} \in Z(GL(\Omega,R/p))$ so that $\overline{X} = \overline{rI}$. Hence $X = rY$, for some $Y \in GL(\Omega,p)$. But then, whenever $\alpha,\beta \in \Omega$, $\alpha \neq \beta$, $Y_{\alpha\beta} \equiv 0 \bmod p$ and $Y_{\alpha\alpha} \equiv 1 \bmod p$. Hence $X_{\alpha\beta} \in p$ and $X_{\alpha\alpha} - X_{\beta\beta} \in p$ which shows that $J(X) \leq p$.

Lemma 3.12. If $X \in Z(GL(\Omega,R))$ then $J(X) = 0$. Moreover, if R is commutative and if $J(Y) = 0$ then $Y \in Z(GL(\Omega,R))$.

Proof. Let $X \in Z(GL(\Omega,R))$, then $X = rI$, for some central unit r of R and so $J(X) = 0$. Conversely, if R is commutative and if $J(Y) = 0$ then Y is a scalar matrix, say $Y = sI$, for some unit $s \in R = Z(R)$. Hence Y is central.

We remark that we can also deduce that when R is commutative and $X \in GL(\Omega,R)$ then $X \in GL'(\Omega,J(X))$. Thus $J(X)$ is the least ideal p of R such that $X \in GL'(\Omega,p)$, when R is commutative. However, when R is not commutative we are not able to draw the same conclusions from knowledge of $J(X)$.

Example 3.1. Let R be the ring of 2×2 matrices over Z and let $X = xI$ where $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $X \in GL(\Omega,R)$ and, since X is diagonal, $J(X) = 0$. Yet X is not in the centre of $GL(\Omega,R)$ since X does not commute with yI where $y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Example 3.2. Again let R be the ring of 2×2 matrices over Z and let $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Let X be the $N \times N$ matrix which has x in every diagonal entry and y in position (1,2). Then $X \in GL(N,R)$. Moreover $J(X)$ is the set of 2×2 matrices all of whose entries are even and, modulo $J(X)$, X is the $N \times N$ matrix that has $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in every diagonal entry and zeros elsewhere. Hence $X \notin GL'(N,J(X))$.

These two examples show that if R is not commutative then $J(X) = 0$ will not guarantee that X is central nor will X necessarily belong to $GL'(\Omega,J(X))$. All we can say is that, modulo $J(X)$, X is diagonal with all the diagonal entries equal.

Given $X \in GL(\Omega,R)$ and $\lambda \in \Omega$ we shall say that the λth column (row) of X is trivial if and only if the λth column (row) of X is equal to the λth column (row) of the identity matrix.

Definition 3.8. Let H be a subgroup of $GL(\Omega,R)$. We define $K(H)$ to be the two sided ideal of R $\Sigma J(X)$, where the sum is taken over all those $X \in H \cap E(\Omega,R)$ that have at least four trivial columns.

Lemma 3.13. If H is a subgroup of $GL(\Omega,R)$ that is normalized by $E(\Omega,R)$ then the following assertions are equivalent:

(i) $H \leq Z(GL(\Omega,R))$,

(ii) $[H,E(\Omega,R)] = 1$,

(iii) $H \cap E(\Omega,R) = 1$.

Proof. Suppose H is central and let $X \in E(\Omega,R) \cap H$. Then X lies in the centre of $E(\Omega,R)$ and so $X = 1$. Hence (i) implies (iii) Since H is normalized by $E(\Omega,R)$, $[H,E(\Omega,R)] \leq H \cap E(\Omega,R)$ so it is clear that (iii) implies (ii). Finally, if (ii) holds, then every X in H commutes with $t(\lambda,x,\mu)$ for all $\lambda,\mu \in \Omega$, $\lambda \neq \mu$ and all $x \in R$. It follows from the proof of Corollary 3.2 that H is central.

In the classification of the normal and subnormal subgroups of $GL(\Omega,R)$, $J(H)$ plays a major part. Our next results show how, in order to simplify the working, we may replace $J(H)$ by what is essentially $K(H)$. Indeed, for normal subgroups H of $GL(\Omega,R)$, we shall show that $J(H) = K(H)$.

Lemma 3.14. Let H be a subgroup of $GL(\Omega,R)$ normalized by $EF(\Omega,p)$ for some two sided ideal p of R. If $A \in H$, then for all $x \in p$, and all $\mu,\rho \in \Omega$, $\mu \neq \rho$, $K(H)$ contains $A_{\mu\rho}x$ and $(A_{\mu\mu} - A_{\rho\rho})x$.

Proof. Let μ, ρ and x be as in the statement of the lemma. We show first that $A_{\mu\rho}x \in K(H)$. This is obvious if $A_{\mu\rho}x = 0$. Thus assume that $A_{\mu\rho}x \neq 0$. Pick $\lambda \in \Omega$, $\lambda \neq \mu, \rho$ and distinct $\varphi_i \in \Omega$, $i = 1,\ldots,4$ such that $\varphi_i \neq \lambda$ and $A^{-1}_{\lambda\varphi_i} = 0$; this is possible since A is column finite. Put $t = t(\lambda,x,\rho)$. If $[t,A] = 1$ then $A_{\alpha\rho}x = 0$, for all $\alpha \neq \rho$ (from the proof of Lemma 3.10), in particular $A_{\mu\rho}x = 0$, contrary to hypothesis. We deduce that t and A do not commute and hence neither do t and A^{-1}. Thus $[t,A^{-1}] \neq 1$. Moreover, since $E(\Omega,R)$ is a normal subgroup of $GL(\Omega,R)$, $[t,A^{-1}] \in E(\Omega,R) \cap H$ and by the choice of φ_i, $[t,A^{-1}]e_{\varphi_i} = e_{\varphi_i}$, $i = 1,\ldots,4$. This shows that for all $\alpha,\beta \in \Omega$, $\alpha \neq \beta$, $[t,A^{-1}]_{\alpha\beta}$ and $([t,A^{-1}] - I)_{\alpha\alpha}$ lie in $K(H)$. Let $t = I + E$. Then

$$[t,A^{-1}] = I - E + AEA^{-1} - EAEA^{-1}.$$

So, for all $\beta \neq \mu$, $[t,A^{-1}]_{\mu\beta} = A_{\mu\rho}xA^{-1}_{\lambda\beta}$ and $([t,A^{-1}] - I)_{\mu\mu} = A_{\mu\rho}xA^{-1}_{\lambda\mu}$. It follows that $K(H)$ contains $\sum_\beta A_{\mu\rho}xA^{-1}_{\lambda\beta}A_{\beta\lambda} = A_{\mu\rho}x$. It remains to show that $K(H)$ contains $(A_{\mu\mu} - A_{\rho\rho})x$. This time we examine the entries that lie on the ρth row of $[t,A^{-1}]$. For any $\beta \neq \lambda, \rho$, $[t,A^{-1}]_{\rho\beta} = A_{\rho\rho}xA^{-1}_{\lambda\beta} - xA_{\lambda\rho}xA^{-1}_{\lambda\beta}$ while $[t,A^{-1}]_{\rho\lambda} = -x + A_{\rho\rho}xA^{-1}_{\lambda\lambda} - xA_{\lambda\rho}xA^{-1}_{\lambda\lambda}$ and $([t,A^{-1}] - I)_{\rho\rho} = A_{\rho\rho}xA^{-1}_{\lambda\rho} - xA_{\lambda\rho}xA^{-1}_{\lambda\rho}$. However, the first part of the proof shows that for any $\alpha,\beta \in \Omega$, $\alpha \neq \beta$, $A_{\alpha\beta}x \in K(H)$. Thus $K(H)$ contains $A_{\rho\rho}xA^{-1}_{\lambda\rho}$, $A_{\rho\rho}xA^{-1}_{\lambda\lambda} - x$ and $A_{\rho\rho}xA^{-1}_{\lambda\beta}$, for any $\beta \neq \lambda, \rho$. In the same way as in the first part we see that $A_{\rho\rho}x - xA_{\lambda\lambda} \in K(H)$ for any $\rho \neq \lambda$. Hence $K(H)$ also contains $A_{\mu\mu}x - xA_{\lambda\lambda}$ and so contains their difference which is $A_{\mu\mu}x - A_{\rho\rho}x$.

Corollary 3.3. Let H be a subgroup of $GL(\Omega,R)$ that is normalized by $EF(\Omega,p)$, for some two sided ideal p of R; then $J(H)p \leq K(H)$.

Proof. If $y \in J(H)p$ then $y = t_1x_1r_1 +\ldots+ t_sx_sr_s$ for x_i a generator of some $J(X_i)$, $t_i \in R$, $r_i \in p$, $i = 1,\ldots,s$ (the X_i need not all be distinct). By Lemma 3.14, $x_ir_i \in K(H)$ and so $y \in K(H)$ since $K(H)$ is a two sided ideal of R.

We remark that Corollary 3.3 shows that when H is normalized by $EF(\Omega,R)$ $J(H) = K(H)$. We shall now deal only with the ideal $K(H)$. We shall show how, given subgroups H of $GL(\Omega,R)$, we can isolate the elementary matrices $t(\lambda,x,\mu)$, for certain $x \in K(H)$ and then use these matrices to construct the elementary subgroups of $GL(\Omega,R)$ that are involved with the classification of the normal and subnormal subgroups of $GL(\Omega,R)$.

Lemma 3.15. Let H be a subgroup of $GL(\Omega,R)$ which is normalized by $EF(\Omega,p)$, for some two sided ideal p of R and let a be a generator of $K(H)$. Then, for all $x_i \in p$, $i = 1,\ldots,4$, and $\rho,\sigma \in \Omega$, $\rho \neq \sigma$, H contains $t(\rho,x_3x_1ax_2x_4,\sigma)$.

Proof. If $a = 0$ then there is nothing to prove, so assume $a \neq 0$. We first suppose that $a = X_{\alpha\beta}$, for some $X \in H \cap E(\Omega,R)$ that has at least four trivial columns, and for some $\alpha,\beta \in \Omega$, $\alpha \neq \beta$. Let Λ index the non-zero entries of the αth row of X and define $f:\Lambda \to R$ by $f(\lambda) = X_{\alpha\lambda}$, if $\lambda \neq \alpha$ and $f(\alpha) = X_{\alpha\alpha} - 1$. There exists $\varphi \in \Omega$, $\varphi \neq \alpha,\rho,\sigma$ such that $X(e_\varphi) = e_\varphi$. Put $t = t(\alpha,x_1,\varphi)$. Then $t_1 = [t,X] = t(\Lambda,x_1f,\varphi)$. If

$\beta \neq \rho$, put $t_2 = t(\beta x_2, \beta)$. Then $t_3 = [t_1, t_2] = t(\rho, x_1 a x_2, \varphi)$. Now put $t_4 = t(\varphi, -x_3, \sigma)$. We see that $t(\rho, x_3 x_1 a x_2, \sigma) = [t_3, t_4] \in H$ since H is normalized by $EF(\Omega, p)$. If $\beta = \rho$, pick $\lambda \neq \beta, \varphi$ and put $t_2 = t(\lambda, x_2, \beta)$. Then $t_3 = [t_1, t_2] = t(\lambda x_1 a x_2, \varphi)$. Now put $t_4 = t(\varphi, -x_3, \sigma)$, so that $t_5 = [t_3, t_4] = t(\lambda, x_3 x_1 a x_2, \sigma)$ and finally, if we put $t_6 = t(\rho, -x_4, \lambda)$ then $[t_6, t_5] = t(\rho x_3 x_1 a x_2 x_4, \sigma) \in H$. We conclude that, in either case, H contains $t(\rho, x_3 x_1 a x_2 x_4, \sigma)$ since in the case $\beta \neq \rho$ we can replace x_2 by $x_2 x_4$.

We next suppose that $a = X_{\alpha\alpha} - 1$, for some $X \in H \cap E(\Omega, R)$ that has at least four trivial columns and for some $\alpha \in \Omega$. With the choice of f, Λ, φ as before, we see that H contains $t_1 = t(\Lambda, x_1 f, \varphi)$. If $\rho \neq \alpha$ then put $t_2 = t(\rho x_2, \alpha)$ and obtain $[t_1, t_2] = t(\rho, x_1 a x_2, \varphi)$ and as before we can deduce that H contains $t(\rho, x_3 x_1 a x_2, \sigma)$. Similarly, if $\rho = \alpha$ then, again as before, we obtain $t(\rho, x_3 x_1 a x_2 x_4, \sigma) \in H$.

Corollary 3.4. Let H be a subgroup of $E(\Omega, R)$ that is normalized by $EF(\Omega, p)$, for some two sided ideal p of R; then H contains $EF(\Omega, p^2 K(H) p^2)$.

Proof. We show first that H contains $t = t(\lambda, y, \mu)$, for all $\lambda, \mu \in \Omega$, $\lambda \neq \mu$ and $y \in p^2 K(H) p^2$. We can write $y = \sum_i x_{3i} x_{1i} a_i x_{2i} x_{4i}$ where $x_{ji} \in p$, $j = 1, \ldots, 4$ and the a_i are generators of $K(H)$. Define $t_i = t(\lambda, x_{3i} x_{1i} a x_{2i} x_{4i}, \mu)$. Then $t_i \in H$ by Lemma 3.15 and $t = \prod_i t_i$ so that $t \in H$. Thus H contains $\{t(\lambda, y, \mu) : \lambda, \mu \in \Omega, \lambda \neq \mu, y \in p^2 K(H) p^2\}$. But, if $Y \in EF(\Omega, R)$, by Lemma 3.1, we see that $K(H) = K(H^Y)$. Moreover, if H is normalized by $EF(\Omega, p)$ then so also is H^Y since

$EF(\Omega,p)$ is a normal subgroup of $EF(\Omega,R)$. In this way we see that $H_0 = \mathrm{Core}_{EF(\Omega,R)}H$ contains $\{t(\lambda,y,\mu) : \lambda,\mu \in \Omega,\ \lambda \neq \mu,\ y \in p^2K(H)p^2\}$. It therefore follows that H_0, and hence H, contain $EF(\Omega,p^2K(H)p^2)$.

Lemma 3.16. Let p be a two sided ideal of R and let $\{p_\alpha\}$ and $\{q_\beta\}$ be two families of finitely generated right ideals whose sums are p. Then

$$\prod_\alpha E(\Omega,p_\alpha) = \prod_\beta E(\Omega,q_\beta).$$

Proof. Let $E_1 = \prod_\alpha E(\Omega,p_\alpha)$ and $E_2 = \prod_\beta E(\Omega,q_\beta)$. It will be sufficient to prove that $E_1 \leq E_2$ since the opposite inclusion will follow by symmetry. Let $X \in E_1$. Then $X = X_1 \ldots X_r$ for $X_i \in E(\Omega,p_{\alpha_i})$, since the $E(\Omega,p_\alpha)$ are normal subgroups. We shall show that each $X_i \in E_2$, for then it will follow that $X \in E_2$. Let $1 \leq i \leq r$. There exist $t_j = t(\Lambda_j,f_j,\mu_j)$, $Y_j \in E(\Omega,R)$, $j = 1,\ldots,s$, such that $X_i = t_1^{Y_1}\ldots t_s^{Y_s}$ and $f_j:\Lambda_j \to p_{\alpha_i}$. But p_{α_i} is finitely generated, say $p_{\alpha_i} = x_1R + \ldots + x_tR$, for some $x_k \in p_{\alpha_i} \leq p$, $k = 1,\ldots,t$. But $p = \Sigma_\beta q_\beta$ so that each x_k belongs to some finite sum of right ideals q_β. We deduce that $p_{\alpha_i} \leq \Sigma_{n_i} q_{\beta_{n_i}}$, where the sum is finite. Thus, by Lemma 3.9, each t_j and hence each X_i belong to the finite product $\prod_{n_i} E(\Omega,q_{\beta_{n_i}}) \leq E_2$. We conclude that $E_1 \leq E_2$.

If p is a two sided ideal of R then p can be expressed as a sum $\Sigma_\alpha p_\alpha$ of finitely generated right ideals - possibly in many ways, for example $p = \Sigma_{x \in p} xR$. Lemma 3.16 shows that the group $\prod_\alpha E(\Omega,p_\alpha)$ is independent of the choice of $\{p_\alpha : \sum_\alpha p_\alpha = p\}$ so we may make the following definition.

Definition 3.9. Let p be a two sided ideal of R and let $\{p_\alpha : \alpha \in A\}$ be a family of finitely generated right ideals of R such that $p = \sum_\alpha p_\alpha$. $E[\Omega,p]$ shall denote the group $\prod_\alpha E(\Omega,p_\alpha)$.

We shall use the group $E[\Omega,p]$ to help classify the normal and subnormal subgroups of $GL(\Omega,R)$, for arbitrary rings with identity.

Lemma 3.17. If H is a subgroup of $GL(\Omega,R)$ normalized by $E(\Omega,R)$ and if H contains $EF(\Omega,q)$, for some two sided ideal q of R, then H contains $E(\Omega,p)$, for any finitely generated right ideal p of R contained in q.

Proof. It will be sufficient to show that H contains any generator $t(\Lambda,f,\mu)$ of $E(\Omega,p)$, since H is normalized by $E(\Omega,R)$. Let $t = t(\Lambda,f,\mu)$ be a generator of $E(\Omega,p)$. Since p is finitely generated, we can write $p = x_1R + \ldots + x_sR$ for $x_i \in p$, $i = 1,\ldots,s$. Then, for each $\lambda \in \Lambda$, $f(\lambda) = \sum_i x_i r_{\lambda i}$. Define $g_i : \Lambda \to R$ by $g_i(\lambda) = r_{\lambda i}$. We can suppose that there exists $\alpha \in \Omega$, $\alpha \notin \Lambda$, $\alpha \neq \mu$ - for if not, we can proceed in two stages, as we did in Lemma 3.7. Put $t_{1i} = t(\Lambda,g_i,\alpha)$ and $t_{2i} = t(\alpha,-x_i,\mu)$. Then $t = \prod_i [t_{1i},t_{2i}]$ and $t \in H$ since H is normalized by $E(\Omega,R)$ and each $t_{2i} \in EF(\Omega,q) \leq H$.

We are now ready to state and prove the theorem which provides the basis for the remaining results of chapter three.

Theorem 3.1. Let G be a subgroup of $GL(\Omega,R)$ containing $E(\Omega,R)$. Let H be a subnormal subgroup of G, say

$$H = H_d \triangleleft H_{d-1} \triangleleft \ldots H_1 \triangleleft H_0 = G.$$

If we put $p = J(H)$ and $q = J(H^G)$ then

$$E[\Omega,p^{f(d)}] \leq H \leq GL'(\Omega,q)$$

where $f(d) = (5^d - 1)/4$ for $d \geq 1$. Moreover, if $d = 1$, if R is commutative or if $H \leq E(\Omega,R)$ then $p = q$.

Proof. If H is a central subgroup then $d = 1$, $J(H) = 0$ and $E(\Omega,0) \leq H \leq GL'(\Omega,0)$. Thus the theorem holds since $J(H) = J(H^G) = 0$. We may thus suppose that H is not central. We shall argue by induction on d. If $d = 1$ then H is a normal subgroup of G and so H is normalized by $E(\Omega,R)$. Corollary 3.4 shows that H contains $EF(\Omega,J(H))$ and Lemma 3.17 shows that H contains $E(\Omega,q)$, for any finitely generated right ideal q contained in $J(H)$. Thus, definition 3.9 shows that H contains $E[\Omega,J(H)]$. We remark that $J(H) \neq 0$, for if not then $J(H \cap E(\Omega,R)) = 0$ and so $H \cap E(\Omega,R) = 1$. Hence, by Lemma 3.13, H is central, contrary to hypothesis. Notice that $J(H)$ is the greatest ideal p such that $E[\Omega,p] \leq H$: for if $x \in p$ then $t(\lambda,x,\mu) \in E(\Omega,xR) \leq E[\Omega,p] \leq H$ so that $x \in J(H)$ and hence $p \leq J(H)$.

Now let $\bar{\ }$ denote images under the homomorphism $GL(\Omega,R) \to GL(\Omega,R/J(H))$ induced by the projection $R \to R/J(H)$. We see that $J(\overline{H}) = 0$ and that $\overline{H \cap E(\Omega,R)} = 1$. By Lemma 3.13 it follows that $\overline{H}$ is central since $\overline{H}$ is normalized by $\overline{E(\Omega,R)}$; we deduce that $H \leq GL'(\Omega,J(H))$.

We now take as inductive hypothesis that the inclusions hold for all subnormal subgroups with normal chains of length less than d. In particular, we see that H_{d-1} contains $E[\Omega,J_0]$, where $J_0 = J(H_{d-1})^{f(d-1)}$.

But then H_d is normalized by $EF[\Omega, J_0]$ so that Corollary 3.4 shows that H contains $EF[\Omega, J_0^2 K(H) J_0^2]$. However, for any $Y \in E(\Omega, R)$ $K(H^Y) = K(H)$ - by Lemma 3.1 - and $H^Y \lhd^d G$ so that $\mathrm{Core}_{E(\Omega,R)} H$ contains $E[\Omega, J_0^2 K(H) J_0^2]$. However Corollary 3.3 shows that $J_0^2 J(H) J_0^3 \leq J_0^2 K(H) J_0^2$ and since $J(H) \leq J(H_{d-1})$ we see that $J(H)^{5f(d-1)+1} \leq J_0^2 K(H) J_0^2$. But $5f(d-1)+1 = f(d)$ and we deduce that H contains $E[\Omega, J(H)^{f(d)}]$.

We next remark that $H \leq H^G \lhd G$ and so $H \leq GL'(\Omega, J(H^G))$, by the inductive basis. Thus with $p = J(H)$ and $q = J(H^G)$ we see that

$$E[\Omega, p^{f(d)}] \leq H \leq GL'(\Omega, q).$$

Finally, we note that, from Lemma 3.1, if R is commutative or if $H \leq E(\Omega, R)$ then $J(H^G) = J(H)$. This completes the proof of the theorem.

Example 3.3. We can show, using Theorem 3.1, that in general it is not the case that $J(X^Y) = J(X)$, for all X, $Y \in GL(\Omega, R)$ (c.f. Lemma 3.1). For, let X be as in example 3.2 and let X_1 be the normal closure of $\{X\}$ in $GL(\Omega, R)$. If $J(X^Y) = J(X)$ then $J(X) = J(X_1)$ and by the theorem $X_1 \leq GL'(\Omega, J(X))$. But then we deduce that $X \in GL'(\Omega, J(X))$, contrary to example 3.2. We conclude that for some $Y \in GL(\Omega, R)$, $J(X^Y) \neq J(X)$.

Definition 3.10. Let p be a two sided ideal of R. We shall say that p is d_R-finite if and only if p can be finitely generated as a right R-module.

Definition 3.11. A ring R is said to be d-finite if and only if each two sided ideal of R is d_R-finite.

For example, simple rings and Noetherian rings are d-finite. However, there are d-finite rings which are neither simple nor Noetherian. Let

R be the ring of $N \times N$ matrices over Q that have only finitely many non-zero entries. R is a simple ring without identity (see, for example, Divinsky, [16]). If we let p_i denote the set of all those $X \in R$ for which only the first i rows have non-zero entries then

$$p_1 < p_2 < \dots < p_i < \dots$$

is an ascending chain of right ideals of R that does not terminate in finitely many steps; thus R is not Noetherian. Now embed R in $R^* = Z \times R$ in the usual way. It follows that R^* is neither Noetherian nor simple. However, any two sided ideal of R^* is finitely generated as a right ideal by $(0,k)$, for some $k \in Z$. Thus R^* is d-finite.

Corollary 3.5. Let R be a d-finite ring and let G be a subgroup of $GL(\Omega,R)$ containing $E(\Omega,R)$. Let H be a subnormal subgroup of G, say

$$H = H_d \triangleleft H_{d-1} \triangleleft \cdots \quad H_1 \triangleleft H_0 = G.$$

If we put $p = J(H)$ and $q = J(H^G)$ then

$$E(\Omega,p^{f(d)}) \leq H \leq GL'(\Omega,q)$$

where $f(d) = (5^d - 1)/4$. Moreover, if $d = 1$, if R is commutative or if $H \leq E(\Omega,R)$ then $p = q$.

Proof. From the proof of Theorem 3.1 we know that

$$EF(\Omega,p^{f(d)}) \leq \mathrm{Core}_{E(\Omega,R)} H \leq H \leq GL'(\Omega,q)$$

and that $p = q$ whenever $d = 1$, R is commutative or $H \leq E(\Omega,R)$. We need only establish that $E(\Omega,p^{f(d)}) \leq H$. However R is d-finite so that $p^{f(d)} = x_1R + \dots + x_sR$, for some $x_i \in p^{f(d)}$. Lemma 3.17 shows that $\mathrm{Core}_{E(\Omega,R)}H$ and hence H contain $E(\Omega,x_iR)$ for $i = 1,\dots,s$ and Lemma 3.9 allows us to deduce that H contains $E(\Omega,p^{f(d)})$.

Theorem 3.2. The following assertions are equivalent.

(i) R is d-finite,

(ii) Whenever H is a subgroup of $GL(\Omega,R)$ normalized by $E(\Omega,R)$ there exists a unique two sided ideal p of R such that

$$E(\Omega,p) \leq H \leq GL'(\Omega,p).$$

Proof. That (i) implies (ii) is immediate from Corollary 3.5. It remains to prove that (ii) implies (i). For any $X \in GL(\Omega,R)$, $X \neq I$, we say that X has finite p-support if there exists a two sided ideal p of R, with $1 \notin p$, and a finitely generated right ideal q of R contained in p such that all the entries of X-I lie in q. Let $X \in E(\Omega,R)$ be a matrix having finite p-support, for some two sided ideal p of R, $1 \notin p$, and let $Y \in E(\Omega,R)$ be an elementary matrix. We assert that $Y^{-1}XY$ has finite p-support. For, let q be a finitely generated right ideal of R giving X finite p-support, say

$$q = \sum_{i=1}^{r} x_i R$$

Let $Y = I + C$ and $X = I + D$. Then

$$Y^{-1}XY = I + D + DC - C(D + DC)$$

since $C^2 = 0$. Moreover, all entries of $D + DC$ lie in q since all the entries of D lie in q; since $D + DC$ and C differ from the identity matrix in only finitely many rows, say in the $\lambda_1,\ldots,\lambda_s$ and μth rows respectively, all the entries of $C(D + DC)$ lie in the right ideal generated by $\{c_{\mu\lambda_k}x_i : 1 \leq k \leq s,\ 1 \leq i \leq r\}$. Hence, all the entries of $Y^{-1}XY - I$ lie in the right ideal generated by $\{x_i,\ c_{\mu\lambda_k}x_i : 1 \leq k \leq s,\ 1 \leq i \leq r\}$, which is contained in p, since p is a two sided ideal.

It now follows inductively that whenever $X \in E(\Omega,R)$ is a matrix having finite p-support and Y is any matrix in $E(\Omega,R)$ then $Y^{-1}XY$ has finite p-support.

We next assert that if q is a finitely generated right ideal contained in a two sided ideal p of R, $1 \notin p$, then every matrix X in $E(\Omega,q)$ has finite p-support. For, let $X = t_1^{X_1} \ldots t_s^{X_s}$, where the t_i are elementary matrices in $E(\Omega,q)$ and the X_i are matrices in $E(\Omega,R)$. Clearly each t_i has finite p-support and the assertion above shows that each $t_i^{X_i}$ has finite p-support. It is clear that the product of any two matrices of finite p-support also has finite p-support and hence our second assertion is established.

Now suppose that (ii) holds yet R is not d-finite. Then there exists a proper two sided ideal p of R which is not finitely generated as a right ideal. Moreover there exists a collection $\{a_i : i \in N\}$ of elements of p that is not contained in any finitely generated right ideal of R that is contained in p. Let p_j denote the right ideal of R that is generated by $\{a_i : 1 \leq i \leq j\}$ and $H = \bigcup_{i \in N} E(\Omega,p_j)$. Then H is a normal subgroup of $E(\Omega,R)$ and our second assertion shows that every matrix in H has finite p-support. By (ii) there exists a unique two sided ideal q of R such that $E(\Omega,q) \leq H \leq GL'(\Omega,q)$. Also $p_i \leq q$, for all $i \in N$, hence $E(\Omega, \bigcup_{i \in N} p_i) \leq H$. Let Ω_0 be a countable proper subset of Ω and $f:\Omega_0 \rightarrow \{a_i : i \in N\}$ be a bijection. Pick $\omega \in \Omega-\Omega_0$ and define $t = t(\Omega_0,f,\omega)$. Then $t \in H$, yet by the choice of the a_i, t does not have finite support. This contradiction completes the proof of the theorem.

Proposition 3.1. Let H be a subgroup of $GL(\Omega,R)$. If, for some two sided ideal p of R, $E(\Omega,p) \leq H \leq GL'(\Omega,p)$, then H is normalized by $E(\Omega,R)$.

Proof. We need to show that $[H,E(\Omega,R)] \leq H$. But

$$[H,E(\Omega,R)] \leq [GL'(\Omega,p),E(\Omega,R)]$$

by hypothesis, so by Lemma 3.8 $[H,E(\Omega,R)] \leq E(\Omega,p) \leq H$.

Theorem 3.2 and Proposition 3.1 correct the work of Maxwell in [27]. We have shown that the commutativity hypothesis is unnecessary but that it is ~~that it is~~ essential to assume that R is d-finite to obtain the usual 'sandwich' type result for normal subgroups of $GL(\Omega,R)$. Theorem 3.1 does show however, that a weaker result producing a sandwiching of normal subgroups is possible without the hypothesis that R is d-finite. Corollary 3.5 produces the infinite dimensional analogue of Wilson's theorem - Theorem 2.2 - and notice that since matrices in $E(\Omega,R)$ differ from the identity matrix in at most finitely many rows, we have 'room to move' and consequently the index function values f(d) are smaller than those produced by Wilson. We shall now construct some examples to show the strengths and the weaknesses of the group $E[\Omega,p]$.

Theorem 3.1 shows that whenever H is a normal subgroup of $GL(\Omega,R)$ H can be sandwiched by the groups $E[\Omega,J(H)]$ and $GL'(\Omega,J(H))$. The advantage of this and other sandwich results is that the groups involved in the sandwich depend only upon the level of H; no other knowledge of H is required. To construct $E[\Omega,J(H)]$ we restricted our attention to the finitely generated right ideals contained in J(H). A reasonable question to ask is whether or not this construction is too restrictive - could we sandwich normal subgroups H of $GL(\Omega,R)$ by groups larger than $E[\Omega,J(H)]$ that also require knowledge of only the level of H. To see that the answer is no, it is useful to consider the following example.

Example 3.4. Let k be a field and R the commutative polynomial ring over k in countably many indeterminates $x_1, x_2,\ldots$ and put $\Omega = N$. For each $j \geq 1$ and $i \geq 1$ define $p_{j,i}$ to be the ideal generated by the indeterminates $x_2,\ldots,x_{2i},x_1,\ldots,x_{2^{j+1}k+1},\ldots,x_{2^{j+1}i+1}$, for $0 < k < i$, and all the indeterminates x_n where n is an odd integer that cannot be expressed as $2^{j+1}p+1$, for any integer p. Thus for each j we have a tower of ideals

$$p_{j,1} < \ldots < p_{j,i} < \ldots$$

where $\cup_i p_{j,i}$ is just the ideal generated by all the indeterminates $x_1, x_2,\ldots.$ Call this ideal p. Given the corresponding tower

$$E(\Omega,p_{j,1}) < \ldots < E(\Omega,p_{j,i}) < \ldots$$

of normal subgroups, we denote by $E(\Omega,p,j)$ the union of all the groups in the tower. Since $E(\Omega,p,j)$ has level p we see that $E[\Omega,p] \leq E(\Omega,p,j)$. It is clear that if $j,k \in N$, $j < k$ then $E(\Omega,p,j) \leq E(\Omega,p,k)$. To see that the inclusion is strict, define the mapping $g:N-\{0\} \to R$ by $g(n) = x_{2^{j+2}n-2^{j+1}+1}$ and the elementary matrix $t = t(N-\{0\},g,0)$. Then $t \in E(\Omega,p,j+1) - E(\Omega,p,j)$.

We shall call the indeterminates $x_2, x_4, \ldots$ even indeterminates and the indeterminates $x_1, x_3, \ldots$ odd indeterminates. If we now reverse the rôles played by the odd and even indeterminates in the construction of the p-chain we obtain the towers $\{q_{j,i} : i \geq 1\}$ and $\{E(\Omega,q,j) : j \geq 1\}$ in a similar way. Since there exist $X \in E(\Omega,p,j)$

such that $J(X)$ is not contained in any ideal generated by all the even indeterminates and only finitely many odd indeterminates, while for all $X \in E(\Omega,q,k)$, $J(X)$ is contained in such ideals, we see that no $E(\Omega,p,j)$ is contained in any $E(\Omega,q,k)$. Similarly we see that no $E(\Omega,q,k)$ is contained in any $E(\Omega,p,j)$. It also follows that since any $X \in E(\Omega,p,j) \cap E(\Omega,q,k)$ must have all its non-trivial entries in some ideal generated by finitely many indeterminates, $X \in E[\Omega,p]$. Thus we have a constructed two infinite towers of normal subgroups of $GL(\Omega,R)$

$$E(\Omega,p,1) < E(\Omega,p,2) < \ldots < E(\Omega,p,j) < \ldots$$

and

$$E(\Omega,q,1) < E(\Omega,q,2) < \ldots < E(\Omega,q,k) < \ldots$$

Every member of each tower has level p and for any j, k

$$E(\Omega,p,j) \cap E(\Omega,q,k) = E[\Omega,p].$$

In this way we see that $E[\Omega,p]$ is 'best possible' with respect to the inclusion of Theorem 3.1.

Proposition 3.1 shows that the sandwich $E(\Omega,p) \leq H \leq GL'(\Omega,p)$ guarantees that H is normalized by $E(\Omega,R)$. Our next example shows that the weaker sandwich involving $E[\Omega,p]$ does not even allow us to conclude that, in general, H is normalized by $EF(\Omega,R)$.

Example 3.5. Let $\Omega = Z$ and let R and p be as in example 3.4. Let H be the subgroup of $E(\Omega,R)$ generated by $E[\Omega,p]$ and $t(Z_+,f,0)$, where $f:Z_+ \to R$ is given by $f(n) = x_n$. Since $t(Z_+,f,-1) = [t(Z_+,f,0),t(0,-1,-1)]$ we see that $t(Z_+,f,-1) \in [EF(\Omega,R),H]$ but $t(Z_+,f,-1) \notin H$.

In our next few results we shall investigate the structure of the lower central series of $E(\Omega,p)$, for two sided ideals p of R.

Lemma 3.18. For any integer $r \geq 1$ and any two sided ideal p of R, $\gamma_{r+1}(E(\Omega,R) \cap GL'(\Omega,p)) \leq E(\Omega,p^r)$.

Proof. We first assert that for any integer $r \geq 1$ and any two sided ideal p of R, $\gamma_r(E(\Omega,R) \cap GL'(\Omega,p)) \leq GL'(\Omega,p^r)$. (Let γ_r denote $\gamma_r(E(\Omega,R) \cap GL'(\Omega,p))$ for brevity.) The inequality clearly holds when $r = 1$, thus suppose that it holds for $r = k$, for some integer $k \geq 1$. Let t_1 be an elementary matrix in γ_k and t_2 be an elementary matrix in γ_1, say $t_1 = I + E_1$ and $t_2 = I + E_2$. Then

$$[t_1,t_2] = I + E_1E_2 + (E_1E_2)^2 - E_2E_1 - E_2E_1E_2 + E_1E_2E_1$$

and since the entries of E_1 and E_2 lie in p^k and p respectively, by our inductive hypothesis, we see that $[t_1,t_2] \in GL'(\Omega,p^{k+1})$. Our assertion now follows. Hence, for any integer $r \geq 1$

$$\gamma_{r+1} = [\gamma_r,\gamma_1] \leq [GL'(\Omega,p^r),E(\Omega,R)] \leq E(\Omega,p^r)$$

by Lemma 3.8.

Lemma 3.19. Let R be a ring and let p be an ideal of R such that p^r is d_R-finite for every integer $r \geq 1$; then $E(\Omega,p^r) \leq \gamma_r(E(\Omega,p))$, for every integer $r \geq 1$.

Proof. Let $r \geq 1$: first notice that from Lemma 3.18 $\gamma_r(E(\Omega,p))$ has level p^r. Next notice that $\gamma_r(E(\Omega,p))$ is normalized by $E(\Omega,R)$ and hence, since p^r is d_R-finite, we see that $E(\Omega,p^r) \leq \gamma_r(E(\Omega,p))$ by Corollary 3.5.

We next show that the inequality of Lemma 3.18 can be strict: we have been unable to ascertain whether or not the same is true of Lemma 3.19.

Example 3.6. Let $\Omega = N$ and $p = (2)$. It follows that, for any integer $r \geq 1$, $p^{r+1} < p^r$ and $J(\gamma_{r+1}(E(\Omega,R) \cap GL'(\Omega,p))) \leq p^{r+1}$ while $J(E(\Omega,p^r)) = p^r$. Moreover, for all $i,j \in \Omega$ and $x \in p^r$, $t(i,x,j)$ is contained in $E(\Omega,p^r)$. Thus $t(i,2^r,j) \in E(\Omega,p^r) - \gamma_{r+1}(E(\Omega,R) \cap GL'(\Omega,p))$ and so we see that the inclusion of Lemma 3.18 can be strict.

Corollary 3.6. For any two sided ideal p of R and any integer $r \geq 1$, $\{E(\Omega,R) \cap GL'(\Omega,p)\}/E(\Omega,p^r)$ is a nilpotent group of class at most r.

Proof. By Lemma 3.18 we see that

$$\gamma_{r+1}(E(\Omega,R) \cap GL'(\Omega,p)) \leq E(\Omega,p^r).$$

But for any integer $k \geq 1$

$$\gamma_k(\{E(\Omega,R) \cap GL'(\Omega,p)\}/E(\Omega,p^r)) \leq \frac{\gamma_k(E(\Omega,R) \cap GL'(\Omega,p))E(\Omega,p^r)}{E(\Omega,p^r)}$$

and hence $\gamma_{r+1}(\{E(\Omega,R) \cap GL'(\Omega,p)\}/E(\Omega,p^r)) = 1$.

Corollary 3.7. Let R be a ring and p a two sided ideal of R such that p^r is d_R-finite, for every integer $r \geq 1$. Whenever $k \leq r$

$$\gamma_k(E(\Omega,p)/E(\Omega,p^r)) = \gamma_k(E(\Omega,p))/E(\Omega,p^r).$$

Proof. This is immediate from the proof of Corollary 3.6 and from Lemma 3.19.

We include here the next lemma and corollary for the sake of completeness although they are not applied anywhere in this chapter. We shall, however find an application for them in chapter four.

Lemma 3.20. Let R be a commutative ring and let p be an ideal of R. Then $[GL'(\Omega,p),GL'(\Omega,p^k)] \leq GL'(\Omega,p^{k+1})$.

Proof. Let $X \in GL'(\Omega,p)$ and $Y \in GL'(\Omega,p^k)$; we shall show that $[X,Y]$ is a scalar matrix modulo p^{k+1}. The result will then follow by Corollary 3.2. First we note that $X = rI + X'$ and $Y = sI + Y'$ where X' and Y' have their entries in p and p^k respectively and r and s are units modulo p and p^k respectively. Moreover, we can write $X^{-1} = tI + X''$ and $Y^{-1} = uI + Y''$ where

$$trI + rX'' + tX' + X''X' = I \qquad \text{and}$$
$$usI + sY'' + uY' + Y''Y' = I.$$

But

$$\begin{aligned}
[X,Y] &= (tuI + uX'' + tY'' + X''Y'')(rsI + sX' + rY' + X'Y') \\
&\equiv tursI + tusX' + ursX'' + usX''X' + trsY'' + turY' + trY''Y' \bmod p^{k+1} \\
&\equiv us(trI + rX'' + tX' + X''X') + tr(sY'' + uY' + Y''Y') \bmod p^{k+1} \\
&\equiv (us + tr(1 - us))I \bmod p^{k+1} \\
&\equiv I \bmod p^{k+1}
\end{aligned}$$

since $tr \equiv 1 \bmod p$ and $us \equiv 1 \bmod p^k$. This completes the proof of the lemma.

Corollary 3.8. Let R be a commutative ring and let p be an ideal of R. Then $GL'(\Omega,p)/GL'(\Omega,p^r)$ is a nilpotent group of class at most r-1.

Proof. We shall examine the lower central series of $GL'(\Omega,p)$.

Since $\gamma_1(GL'(\Omega,p)) = GL'(\Omega,p)$ we take as inductive hypothesis that $\gamma_k(GL'(\Omega,p)) \leq GL'(\Omega,p^k)$, for all integers $k < r$. But

$$\begin{aligned}\gamma_r(GL'(\Omega,p)) &= [\gamma_{r-1}(GL'(\Omega,p)),GL'(\Omega,p)] \leq [GL'(\Omega,p^{r-1}),GL'(\Omega,p)] \\ &\leq GL'(\Omega,p^r)\end{aligned}$$

by the lemma. We now see that $\gamma_r(GL'(\Omega,p)/GL'(\Omega,p^r)) = 1$ and this completes the proof of the corollary.

Proposition 3.2. Let R be a d-finite ring and let H be a subgroup of $E(\Omega,R)$. The following assertions are equivalent.

(i) H is a subnormal subgroup of $E(\Omega,R)$

(ii) for some unique two sided ideal p of R and some integer m

$$E(\Omega,p^m) \leq H \leq GL'(\Omega,p).$$

Moreover, if (i) holds the least m in (ii) satisfies $d - 1 \leq m \leq f(d)$. If (ii) holds then the defect of H in $E(\Omega,R)$ is at most $m + 1$. (f is defined in the statement of Theorem 3.1.)

Proof. If (i) holds then (ii) follows from Corollary 3.5 and Lemma 3.1, since $J(H^Y) = J(H)$ for all $Y \in GL(\Omega,R)$. Now suppose that (ii) holds. Since $E(\Omega,p)/E(\Omega,p^m)$ is nilpotent there is a series

$$E(\Omega,p^m) \leq \gamma_m \leq \dots \leq \gamma_1 \leq E(\Omega,R)$$

where $\gamma_i = \gamma_i(E(\Omega,p))$, $i = 1,\dots,m$. But

$$[E(\Omega,R),\gamma_1 H] \leq E(\Omega,p)[E(\Omega,R),GL'(\Omega,p)] \leq E(\Omega,p)$$

shows that

$$H \lhd \gamma_m H \lhd \dots \lhd \gamma_1 H \lhd E(\Omega,R)$$

is a normal series from H to $E(\Omega,R)$ of length $m + 1$ and we conclude that H is a subnormal subgroup of $E(\Omega,R)$ of defect at most $m + 1$.

It remains to show that if (i) holds then the least m in (ii) satisfies $d - 1 \leq m \leq f(d)$. It is clear that $m \leq f(d)$ ~~since we can~~

since we can always take $m = f(d)$, although a lesser power of p may suffice. The proof of (ii) implies (i) shows that if (ii) holds then $d \leq m + 1$, that is $d - 1 \leq m$. This completes the proof of the proposition.

Proposition 3.2 shows that subgroups of $E(\Omega,R)$ are subnormal if and only if they are sandwiched by the groups $E(\Omega,p^m)$ and $GL'(\Omega,p)$, for some two sided ideal p and some integer m. It also shows that $d - 1 \leq m \leq f(d)$, where d is the defect of the subnormal subgroup. Clearly, for large d, there is a large discrepancy between $d - 1$ and $f(d)$ and we have not been able to find an example to show that it may be necessary to take $m = f(d)$. However, our next example shows that we can, on occasion, take $m = d$.

Example 3.7. For any ring R and subring S of R, define the subideals S_i of R by $S_0 = R$, $S_i = S^{S_{i-1}}$, for $i \geq 1$ (S^R denotes the ideal of R generated by S). This definition gives rise to a descending chain of subideals

$$S_i \lhd \ldots \lhd S_2 \lhd S_1 \lhd S_0 = R$$

of R. For $i \geq 1$, $j \geq 1$, write $(S_i)_j = S_{i,j}$. We assert that $S_{i,j} = S_j$ whenever $j \leq i$. For, certainly $S_j \leq S_{i,j}$ since $S \leq S_i$. Hence

$$S_{i,j} = S_i^{S_{j-1}} \leq S^{S_{j-1}} \leq S_j$$

since S_j is an ideal of S_{j-1}.

Choose $R = Z[x]$ and let S be the subring of R generated by x^2. Since $x^{2i-1} \in S_{i-1} - S_i$, for all $i \geq 1$, we see that $S_i < S_j$ whenever $i > j$. We assert that S_i has defect i in R, for if not then $S_i \lhd^{i-1} R$ and hence $S_{i,i-1} = S_i$ and this is contrary to $S_i < S_{i-1}$.

Denote by $T\langle S_i\rangle$ the set $\{t(\Lambda,f,\mu) : \Lambda \subset \Omega,\ \mu \in \Omega-\Lambda,\ f:\Lambda \to S_i\}$. Define, for $i \geq 2$, $E\langle S_i\rangle$ to be the normal closure of $T\langle S_i\rangle$ in $E\langle S_{i-1}\rangle$, where $E\langle S_1\rangle = E(\Omega,S_1)$. These definitions give rise the the chain

$$\ldots \triangleleft E\langle S_i\rangle \triangleleft \ldots \triangleleft E\langle S_1\rangle \triangleleft E\langle S_0\rangle = E(\Omega,R)$$

of subnormal subgroups of $E(\Omega,R)$. It is clear that, for each $i \geq 1$ and all $g \in E(\Omega,R)$, $J(E\langle S_i\rangle^g) = J(E\langle S_i\rangle) = S_1$ and that $S_1^i \leq S_i$. Hence $T\langle S_1^i\rangle$ is contained in $\mathrm{Core}_{E(\Omega,R)} E\langle S_i\rangle$ and since R is d-finite, applying Lemma 3.17 we see that $E(\Omega,S_1^i) \leq E\langle S_i\rangle$. We now assert that $E\langle S_i\rangle$ has defect i in $E(\Omega,R)$. If the defect of $E\langle S_i\rangle$ is less than i then there is a series

$$E\langle S_i\rangle \triangleleft H_{i-2} \triangleleft \ldots \triangleleft H_1 \triangleleft E(\Omega,R)$$

from $E\langle S_i\rangle$ to $E(\Omega,R)$. But this produces a chain of subideals from S_i to R as follows: let $T_1 = J(H_1)$ and let T_j be the subideal of T_{j-1} generated by all the ring elements $(X - I)_{\alpha\beta}$, for all $\alpha,\beta \in \Omega$ and $X \in H_j$. Then $T_{i-1} = S_i$ and so we have $S_i \triangleleft^{i-1} R$, contrary to the choice of i. Thus, if we put $H = E\langle S_i\rangle$ we see that H is a subnormal subgroup of $E(\Omega,R)$ of defect i with $E(\Omega,J(H)^i) \leq H \leq GL'(\hat{\Omega},J(H))$.

It is clear that simple rings are d-finite and so Proposition 3.2 shows that whenever R is a simple ring with identity $E(\Omega,R)$ is a simple group. If we now consider rings that do not have identities then simple rings without identity need not necessarily be d-finite and $E(\Omega,R)$ need not necessarily be simple . In fact we have

Proposition 3.3. Let R be a simple ring without identity with $R^2 \neq 0$. If R is d-finite then $E(\Omega,R)$ is simple. If R is not d-finite then $E(\Omega,R)' < E(\Omega,R)$ and $E(\Omega,R)'$ is simple and is the unique minimal normal subgroup of $E(\Omega,R)$.

Proof. Suppose that R is d-finite and simple. Embed R in $R^* = Z \times R$ in the usual way; it follows that R^* is d-finite. We can consider $E(\Omega,R)$ as a normal subgroup of $E(\Omega,R^*)$ and whenever H is a non-trivial normal subgroup of $E(\Omega,R)$, $H \triangleleft E(\Omega,R) \triangleleft E(\Omega,R^*)$. By Corollary 3.5, we see that $E(\Omega,J(H)^6) \leq H$. But $J(H) = R$ and $R^2 \neq 0$ so that $J(H)^6 = R$ and hence $H = E(\Omega,R)$.

Suppose now that R is not d-finite but is simple and that H is a non-trivial normal subgroup of $E(\Omega,R)$. Then, as before, by embedding R in $R^* = Z \times R$ and by Corollary 3.4 we see that H contains $EF(\Omega,R)$. Let $t_1 = t(\Lambda_1,f,\mu)$ and $t_2 = t(\Lambda_2,g,\rho)$. We shall show that H contains $[t_1,t_2]$ and, since H is a normal subgroup of $E(\Omega,R)$, it will follow that H contains $E(\Omega,R)'$ and that $E(\Omega,R)'$ is the unique minimal normal subgroup of $E(\Omega,R)$. If $\mu \notin \Lambda_2$ then $[t_1,t_2] = t(\Lambda_2,f(\rho)g,\mu) = [t(\rho,f(\rho),\mu), t(\Lambda_2,g,\rho)]$ or if $\rho \notin \Lambda_1$ then $[t_1,t_2] = t(\Lambda_1,-g(\mu)f,\rho) = [t(\Lambda_1,f,\mu), t(\mu,g(\mu),\rho)]$. In each case we see that $[t_1,t_2] \in H$ since $EF(\Omega,R) \leq H$. Suppose therefore that $\mu \in \Lambda_2$ and $\rho \in \Lambda_1$. Then write $t_1 = t_1't_1''$ and $t_2 = t_2't_2''$ where $t_1'' = t(\rho,f(\rho),\mu)$ and $t_2'' = t(\mu,g(\mu),\rho)$. Then

$$[t_1,t_2] = [t_1',t_2'']^{t_1''}[t_1'',t_2''][t_1',t_2']^{t_1''t_2''}[t_1'',t_2']^{t_2''}$$

and hence $[t_1,t_2] \in H$ by the remarks above and since $EF(\Omega,R) \leq H$.

This therefore shows that $E(\Omega,R)'$ is the unique minimal normal subgroup of $E(\Omega,R)$. In particular $E(\Omega,R)' = E[\Omega,R]$. Now let H be a non-trivial normal subgroup of $E(\Omega,R)'$. Then $J(H) = R$ and by Theorem 3.1 we see that $E[\Omega,R] \leq H$, that is $H = E(\Omega,R)'$; we conclude that $E(\Omega,R)'$ is simple.

Finally notice that the proof of theorem 3.2 shows that every $X \in E(\Omega,R)'$ has finite R-support while there exists $X \in E(\Omega,R)$ that do not have finite R-support. We conclude that $E[\Omega,R] = E(\Omega,R)'$ is simple and is the unique minimal proper normal subgroup of $E(\Omega,R)$. Moreover, $E(\Omega,R)'$ is just the subgroup comprising all those $X \in E(\Omega,R)$ that have finite R-support.

Proposition 3.3 deals with the simplicity of $E(\Omega,R)$ and indeed shows, together with Corollary 3.1 and Lemmas 3.18 and 3.19 that when R is simple (with or without identity) $E(\Omega,R)$ is perfect if and only if $R^2 = R$ and R is d-finite. In fact we are able to make the stronger statement that for any two sided ideal p of a ring R with identity, $E(\Omega,p)$ is perfect if and only if $p^2 = p$ and p is d_R-finite. For, if p is finitely generated as a right ideal and if $p^2 = p$, Lemma 3.19 shows that

$$E(\Omega,p) = E(\Omega,p^2) \leq \gamma_2(E(\Omega,p)) \leq E(\Omega,p).$$

It follows that $E(\Omega,p)$ is perfect. Now suppose that either $p^2 \neq p$ or p is not d_R-finite. If $p^2 < p$, pick $x \in p-p^2$. It follows that $t(\lambda,x,\mu) \in E(\Omega,p)-\gamma_2(E(\Omega,p))$, for any $\lambda,\mu \in \Omega$, $\lambda \neq \mu$, since $\gamma_2(E(\Omega,p))$ has level p^2 while $E(\Omega,p)$ has level p. If p is not d_R-finite then the proof of Proposition 3.3 shows that every matrix in $E(\Omega,p)'$ has finite p-support while the proof of Theorem 3.2 shows that there exist matrices in $E(\Omega,p)$ that do not have finite p-support. In either case $E(\Omega,p)$ is not perfect.

Proposition 3.3 also shows that for rings without identity, the factor group $E(\Omega,R)/E(\Omega,R)'$ depends very much upon the way in which R is generated as a right R-module. Our next example shows just how far from trivial this factor group can be.

Example3.8. For any ordinal α we can choose R and Ω such that there are α normal subgroups between $E(\Omega,R)'$ and $E(\Omega,R)$. We show first how to construct the ring R.

Let $(\Lambda, \leq)$ be a well ordered set with card $\Lambda = \aleph_\alpha$. Let V be the free k-module $k^{(\Lambda)}$, where k is a field and let $M_\alpha(k)$ denote the endomorphism ring of V. For each $X \in M_\alpha(k)$ define the rank of X, $\rho(X)$ as the k-dimension of the image space of X, For each ordinal β, $0 \leq \beta \leq \alpha$ define the sets

$$N_\beta = \{X : X \in M_\alpha(k),\ \rho(X) < \aleph_\beta\}.$$

We see that each N_β is a proper ideal of $M_\alpha(k)$, (for example, [8]) and that

$$N_0 < N_1 < \ldots < N_\beta < \ldots$$

is an ascending chain of ideals in $M_\alpha(k)$. We also see from [8] that $N_{\beta+1}/N_\beta$ are simple rings without identity, for each ordinal β, $0 \leq \beta \leq \alpha$.

For any ring R we define the cardinal d(R) to be the least cardinal $\underline{u}$ amongst those cardinals $\underline{v}$ such that R is generated as a right R-module by a set of cardinality $\underline{v}$. For example, if R is a d-finite ring then $d(R) < \aleph_0$ or if $R = N_0$ then $d(R) = \aleph_0$. We now assert that for each ordinal β, $0 \leq \beta < \alpha$, $d(N_{\beta+1}/N_\beta) = \aleph_{\beta+1}$. To do this it will be sufficient to show that we can construct an ascending chain of right ideals in $N_{\beta+1}/N_\beta$ which does not terminate in $\aleph_\beta$ steps.

We now abbreviate $N_{\beta+1}/N_\beta$ to $\overline{N}_{\beta+1}$ and let ω_β be the first ordinal of cardinality $\aleph_\beta$. Define the set S of ordinals by

$$S = \{\gamma : \gamma \text{ is an ordinal, card } \gamma \leq \aleph_\beta\}.$$

We next define the set $S(\omega_\beta)$ by

$$S(\omega_\beta) = \{\omega_\beta \gamma : \gamma \in S\}.$$

We see that $S(\omega_\beta)$ is a tower of ordinals, each of cardinality $\aleph_\beta$, which does not stop in $\aleph_\beta$ steps, since, if γ is an ordinal of cardinality $\aleph_\beta$ then so also is $\gamma+1$ and $\omega_\beta\gamma < \omega_\beta(\gamma+1)$.

Let $\gamma \in S$ and define P_γ to be the set of all those $X \in M_\alpha(R)$ for which the image of X is the direct sum of at most the first $\omega_\beta\gamma$ copies of k. Then whenever $\gamma,\delta \in S$, $\gamma < \delta$, $\overline{P}_\gamma \neq 0$ and $\overline{P}_\gamma < \overline{P}_\delta$. Hence $\{\overline{P}_\gamma : \gamma \in S\}$ is an ascending chain of right ideals of $\overline{N}_{\beta+1}$ which does not terminate in $\aleph_\beta$ steps. We deduce that $d(\overline{N}_{\beta+1}) = \aleph_{\beta+1}$. Thus, for any ordinal $\beta \geq 0$ we have established the existence of simple rings R without identity for which $d(R) = \aleph_{\beta+1}$.

Let Ω be a set of cardinality $\aleph_\alpha$ and let R be a simple ring without identity with $d(R) = \aleph_{\alpha+1}$. Let $X \in E(\Omega,R)$ and $\underline{c}$ be the cardinality of a minimal generating set for the right ideal generated by $X_{\alpha\beta}$, $X_{\alpha\alpha} - X_{\beta\beta}$ $(\alpha,\beta \in \Omega,\ \alpha \neq \beta)$ and let $\aleph_\beta$ be the first infinite cardinal greater than $\underline{c}$. We shall say that X has $\aleph_\beta$-support. Let

$$E(\beta) = \langle X : X \in E(\Omega,R),\ X \text{ has } \aleph_\xi\text{-support},\ \xi \leq \beta \rangle^{E(\Omega,R)}$$

The proof of Theorem 3.2 shows that $E(\beta)$ is a proper normal subgroup of $E(\Omega,R)$ whenever $\beta \leq \alpha$ and that

$$\{E(\gamma) : 0 \leq \gamma \leq \alpha+1\}$$

is a tower of normal subgroups of $E(\Omega,R)$ with $E(0) = E(\Omega,R)'$ and $E(\alpha+1) = E(\Omega,R)$, since Ω has cardinality $\aleph_\alpha$. Hence there are α distinct normal subgroups between $E(\Omega,R)$ and $E(\Omega,R)'$.

We end chapter three with some miscellaneous results concerning subnormal subgroups of $E(\Omega,R)$.

Corollary 3.9. If R is a residually nilpotent ring then $E(\Omega,R)$ is a residually nilpotent group.

Proof. Let γ_i denote $\gamma_i(E(\Omega,R))$ and γ_ω denote $\bigcap_{i=1}^{\infty} \gamma_i$. From the proof of Lemma 3.18 we see that $\gamma_i \leq GL'(\Omega,R^i)$. We assert that

$\bigcap_i GL'(\Omega,R^i) \leq Z(GL'(\Omega,R))$. For, if $X \in GL'(\Omega,R^i)$ then $X \equiv rI \bmod R^i$, for some $r \in R$. Thus if $X \in \bigcap_i GL'(\Omega,R^i)$ we see that all the off-diagonal entries lie in $\bigcap_i R^i = 0$, that is $X = rI$, for some $r \in R$. Moreover, if $X \in GL'(\Omega,R^i)$ then for all $x \in R$, $rx-xr \in R^i$. Thus, if $X \in \bigcap_i GL(\Omega,R^i)$ we see that r is a central unit and $X \in Z(GL(\Omega,R))$. Thus $\bigcap_i \gamma_i \leq Z(GL(\Omega,R))$. But, since $E(\Omega,R)$ has trivial centre we deduce that $\bigcap_i \gamma_i = 1$ and that $E(\Omega,R)$ is residually nilpotent.

Definition 3.12. Let R be a commutative ring without identity. We shall say that R is a finitely generated scalar residually nilpotent domain if and only if

(i) R is residually nilpotent

(ii) R^n is finitely generated for each integer $n \geq 1$

(iii) every principal ideal of R contains R^i for some integer $i \geq 0$.

If R is a discrete valuation ring and m is its maximal ideal then, since every ideal of R is principal and is just a power of m, we see that m is a finitely generated scalar residually nilpotent domain (c.f. Theorem 1.3).

Proposition 3.4. Let R be a finitely generated scalar residually nilpotent domain. Every subnormal subgroup of $E(\Omega,R)$ is a $\overline{Z}$ group.

Proof. First notice that $E(\Omega,R)$ is residually nilpotent. Let H be a subnormal subgroup of $E(\Omega,R)$ and let J denote the level of H; we may suppose that H is not nilpotent and so $J^n \neq 0$, for all $n \geq 1$. By

Corollary 3.5, H contains $E[\Omega,J^i]$, for some integer i. Since H is non-central, $J \neq 0$ and so $J^i \neq 0$, by our supposition. Pick $x \in J$, $x \neq 0$ and put $y = x^i$. By hypothesis $R^n \leq (y) \leq J^i$ and hence $E(\Omega,R^n) \leq H$ since R^n is finitely generated. To show that H is a $\overline{Z}$ group it will be sufficient to show that whenever K is a non-trivial normal subgroup of H there exists a normal subgroup L of H with $L \leq K$ and H/L nilpotent. But if K is a non-trivial normal subgroup of H it will also be a non-trivial subnormal subgroup of $E(\Omega,R)$. Hence, for some integer m, $E(\Omega,R^m) \leq K$. If we take $L = E(\Omega,R^m)$ we see that H/L is a subgroup of $E(\Omega,R)/E(\Omega,R^m)$ which, by Corollary 3.6, is nilpotent. Hence H/L is nilpotent and this completes the proof of the proposition.

We now conclude chapter three with a result concerning the join of subnormal subgroups of $E(\Omega,R)$.

Proposition 3.5. Let R be a d-finite ring.

(i) If $H_i \triangleleft^{d_i} E(\Omega,R)$, for $i = 1,\ldots,r$ then $\langle H_1,\ldots,H_r\rangle \triangleleft^d E(\Omega,R)$ where $d = \sum_{i=1}^{r} f(d_i) + 1$.

(ii) If $\{H_\alpha : \alpha \in A\}$ is a family of subnormal subgroups of $E(\Omega,R)$ then $\langle H_\alpha : \alpha \in A\rangle$ is also a subnormal subgroup of $E(\Omega,R)$.

Proof. (i) Let J_i denote the level of H_i, $i = 1,\ldots,r$. Then by Corollary 3.5, each H_i contains $E(\Omega,J_i^{f(d_i)})$. But $J(H_i) = J(H_i^{E(\Omega,R)})$ and since $E(\Omega,R) \cap GL'(\Omega,J_i)/E(\Omega,J_i^{f(d_i)})$ is nilpotent of class at most $f(d_i)$ and $H_i^{E(\Omega,R)} \leq E(\Omega,R) \cap GL'(\Omega,J_i)$ we see that

$\gamma_{f(d_i)}(H_i^{E(\Omega,R)}) \leq E(\Omega, J_i^{f(d_i)}) \leq H_i$. Thus, by Theorem 1.1, we see that $\langle H_1, \ldots, H_r \rangle$ is subnormal of defect at most $\sum_{i=1}^{r} f(d_i) + 1$.

(ii) For each $\alpha \in A$ write $J_\alpha = J(H_\alpha)$ and $n_\alpha = f(d_\alpha)$. Let J be the sum of the ideals J_α, for $\alpha \in A$ and K be the sum of the ideals $J_\alpha^{n_\alpha}$, for $\alpha \in A$. Since R is d-finite, $J^r \leq K$, for some integer r. Hence

$$E(\Omega, J^r) \leq E(\Omega, K) \leq \langle E(\Omega, J^{n_\alpha}) : \alpha \in A \rangle \lneq \langle H_\alpha : \alpha \in A \rangle \leq GL'(\Omega, J)$$

and we conclude that $\langle H_\alpha : \alpha \in A \rangle$ is subnormal in $E(\Omega,R)$ by applying Proposition 3.2.

We have not been able to decide if it is necessary to assume here that R is d-finite.

Chapter four

In chapter three we classified the normal and subnormal subgroups H of $GL(\Omega,R)$ in terms of $GL'(\Omega,J(H))$ and $E(\Omega,J(H))$. We shall now show that very similar results are possible when classifying the normal and subnormal subgroups of $Sp(\Omega,R)$. We shall define a normal subgroup $ESp(\Omega,R)$ corresponding to $E(\Omega,R)$ and for subnormal subgroups H of $Sp(\Omega,R)$ we shall see that the level of H will play a major rôle in the classification.

Maxwell has studied the normal subgroup structure of $Sp(\Omega,R)$ in [28] and there he defines a group which is denoted by $ESp(R)$. We shall see that our group $ESp(\Omega,R)$, although defined differently, coincides with $ESp(R)$ and we shall thus extend Maxwell's classification to cover subnormal subgroups of $Sp(\Omega,R)$. The methods we shall use are based on those of Robertson [33] and Wilson [40].

Unless otherwise stated, throughout this chapter R shall denote a ring with identity and Ω shall be an infinite set.

Definition 4.1. Let $X \in Sp(\Omega,R)$. We shall denote by $J(X)$ the two sided ideal of R generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha}-X_{\beta\beta}$, for all $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$. We shall call $J(X)$ the level of X. (Ω_1 is as defined in chapter one.)

We see that the level of X, regarded as a matrix in $Sp(\Omega,R)$ is the same as the level of X regarded as a matrix in $GL(\Omega_1,R)$.

Definition 4.2. Let $H \leq Sp(\Omega,R)$. We shall denote by $J(H)$ the

the two sided ideal $\sum_{X \in H} J(X)$. We shall call $J(H)$ the level of H.

We see that the level of H, regarded as a subgroup of $Sp(\Omega,R)$, is the same as the level of H regarded as a subgroup of $GL(\Omega_1,R)$.

Lemma 4.1. Let R be a commutative ring.

(i) Forall $X, Y \in Sp(\Omega,R)$, $J(X^Y) = J(X)$.

(ii) For $H \leq Sp(\Omega,R)$ and $X \in Sp(\Omega,R)$, $J(H^X) = J(H)$.

Proof (i) Since $X, Y \in GL(\Omega_1,R)$ it follows from Lemma 3.1 that $J(X^Y) = J(X)$. (ii) Since $H \leq GL(\Omega_1,R)$ and $X \in GL(\Omega_1,R)$ it follows from Lemma 3.1 that $J(H^X) = J(X)$.

The classification of the normal and subnormal subgroups of $Sp(\Omega,R)$ that we shall give here is based on the construction of a group analogous to the group $E(\Omega,R)$. To make this construction we need to make the following definition.

Definition 4.3. Let $a \in R$ and $\alpha,\beta \in \Omega_1$. We denote the $\Omega_1 \times \Omega_1$ matrix whose only non-zero entry is 1 in the (α,β)th position by $e_{\alpha\beta}$. We shall also use 1 to denote the $\Omega_1 \times \Omega_1$ identity matrix. Let $\lambda,\mu \in \Omega$, $\lambda \neq \mu$ and $a \in R$. Define the $\Omega_1 \times \Omega_1$ matrices $t_{\lambda\mu}(a)$, $r_{\lambda\mu}(a)$, $r_{\lambda\lambda}(a)$, $s_{\lambda\mu}(a)$ and $s_{\lambda\lambda}(a)$ by:

$$t_{\lambda\mu}(a) = 1 + a(e_{\lambda\mu} - e_{\mu'\lambda'})$$
$$s_{\lambda\mu}(a) = 1 + a(e_{\lambda'\mu} + e_{\mu'\lambda})$$
$$s_{\lambda\lambda}(a) = 1 + ae_{\lambda'\lambda}$$
$$r_{\lambda\mu}(a) = 1 + a(e_{\lambda\mu'} + e_{\mu\lambda'})$$
$$r_{\lambda\lambda}(a) = 1 + ae_{\lambda\lambda'}$$

These definitions are the same as those given in [33]. It is clear that whenever $\lambda,\mu \in \Omega$ and $a \in R$, $r_{\lambda\mu}(a)r_{\lambda\mu}(b) = r_{\lambda\mu}(a+b)$ and $s_{\lambda\mu}(a)s_{\lambda\mu}(b) = s_{\lambda\mu}(a+b)$. We also quote from [33] the following lemma which is a list of commutator identities involving the matrices given in definition 4.3.

Lemma 4.2. Let R be a commutative ring, a, b $\in$ R, and $\lambda,\mu,\alpha,\beta \in \Omega$.

(i) $[r_{\lambda\mu}(b), s_{\alpha\beta}(a)] = 1$, λ,μ distinct from α,β,

(ii) $[r_{\lambda\alpha}(b), s_{\alpha\beta}(a)] = t_{\lambda\beta}(ba)$ α,β,λ distinct,

(iii) $[r_{\lambda\alpha}(b), s_{\alpha\alpha}(a)] = t_{\lambda\alpha}(ba)r_{\lambda\lambda}(b^2a)$ $\lambda \neq \alpha$,

(iv) $[r_{\lambda\lambda}(b), s_{\alpha\lambda}(a)] = t_{\lambda\alpha}(ba)s_{\alpha\alpha}(-ba^2)$ $\alpha \neq \lambda$,

(v) $[s_{\lambda\mu}(a), t_{\lambda\beta}(b)] = s_{\mu\beta}(ab)$ λ,μ,β distinct,

(vi) $[s_{\lambda\mu}(a), t_{\alpha\mu}(b)] = 1$ $\alpha \neq \lambda,\mu$,

(vii) $[s_{\lambda\mu}(a), t_{\lambda\mu}(b)] = s_{\mu\mu}(2ab)$ $\lambda \neq \mu$,

(viii) $[s_{\lambda\lambda}(a), t_{\lambda\mu}(b)] = s_{\lambda\mu}(ab)s_{\mu\mu}(ab^2)$ $\lambda \neq \mu$,

(ix) $[s_{\lambda\mu}(a), t_{\alpha\beta}(b)] = 1$ λ,μ distinct from α,β,

(x) $[r_{\lambda\mu}(a), t_{\alpha\beta}(b)] = 1$ λ,μ distinct from α,β,

(xi) $[r_{\lambda\mu}(a), t_{\alpha\mu}(b)] = r_{\alpha\lambda}(-ab)$ λ,μ,α distinct,

(xii) $[r_{\lambda\mu}(a), t_{\lambda\beta}(b)] = 1$ $\beta \neq \lambda,\mu$,

(xiii) $[r_{\lambda\mu}(a), t_{\lambda\mu}(b)] = r_{\lambda\lambda}(-2ab)$ $\lambda \neq \mu$,

(xiv) $[r_{\lambda\lambda}(a), t_{\alpha\lambda}(b)] = r_{\alpha\lambda}(-ab)r_{\alpha\alpha}(ab^2)$ $\lambda \neq \alpha$.

Although we have taken as hypothesis in Lemma 4.2 that R is commutative, it is clear that the commutator identities will still hold if R is not commutative but one of a or b is the identity of R.

Definition 4.4. $ESp(\Omega,R)$ is defined to be the subgroup of $Sp(\Omega,R)$

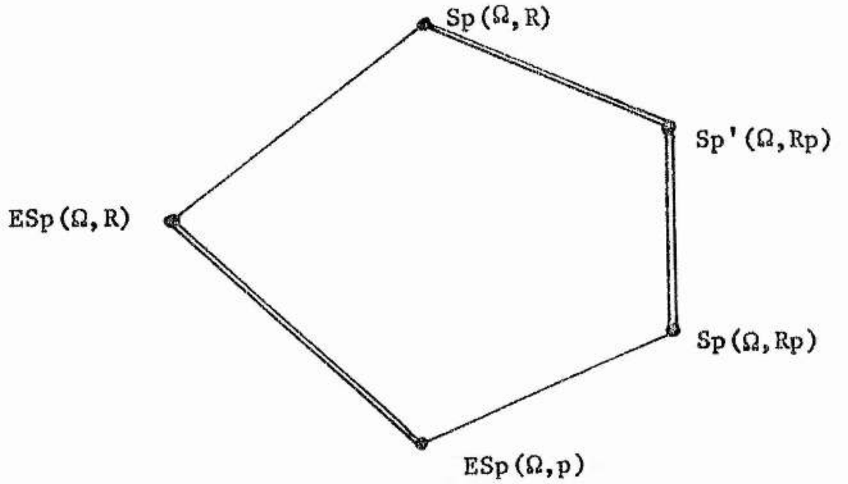
generated by $\{r_{\lambda\mu}(a), s_{\lambda\mu}(a) : \lambda, \mu \in \Omega,\ a \in R\}$.

Definition 4.5. For any right ideal p of R, $ESp(\Omega, p)$ is defined to be the normal closure of $\{r_{\lambda\mu}(a),\ s_{\lambda\mu}(a) : \lambda, \mu \in \Omega,\ a \in p\}$ in $ESp(\Omega, R)$.

Lemma 4.3. Let p be a right ideal of R; $ESp(\Omega, p) \leq Sp(\Omega, Rp)$.

Proof. It is clear that, for all $\lambda, \mu \in \Omega$ and $x \in p$, $r_{\lambda\mu}(x)$, $s_{\lambda\mu}(x) \in Sp(\Omega, Rp)$. It follows that $ESp(\Omega, p) \leq Sp(\Omega, Rp)$ since $Sp(\Omega, Rp)$ is a normal subgroup of $Sp(\Omega, R)$.

We now see that the subgroups $ESp(\Omega, R)$ and $ESp(\Omega, p)$ satisfy the relations implied by the following diagram.



(A single line denotes subgroup inclusion, a double line denotes normal subgroup inclusion.) We shall use the groups $ESp(\Omega, p)$ and $Sp'(\Omega, p)$ to sandwich the normal and subnormal subgroups of $Sp(\Omega, R)$. We have defined the groups $ESp(\Omega, R)$ and $ESp(\Omega, p)$ in terms of the matrices $r_{\lambda\mu}(a)$ and $s_{\lambda\mu}(a)$ for $\lambda, \mu \in \Omega$ whereas, in Theorems 2.7 - 2.9 the 'elementary' subgroups are generated in terms of the symplectic transvections $t(a,x)$, for unimodular

$x \in R^{(\Omega_1)}$. Our next result shows that these definitions are equivalent so that we may choose the generating set of $ESp(\Omega,R)$ to be either $\{r_{\lambda\mu}(a),\ s_{\lambda\mu}(a) : \lambda,\mu \in \Omega,\ a \in R\}$ or $\{t(a,x) : a \in R,\ x$ a unimodular element of $R^{(\Omega_1)}\}$, whichever is the more convenient at the time. Recall that we may construct the alternate bilinear form $(*,*)$ given the $\Omega_1 \times \Omega_1$ matrix J and the canonical basis $\{e_\lambda, e_{\lambda'} : \lambda \in \Omega\}$ for $R^{(\Omega_1)}$ and that for any $a \in R$ and unimodular $x \in R^{(\Omega_1)}$ the symplectic transvection $t(a,x)$ is defined by

$$t(a,x)m = m + xa(m,x), \qquad m \in R^{(\Omega_1)}.$$

Let $G(R)$ be the group generated by the $t(a,x)$, for all $a \in R$ and all unimodular $x \in R^{(\Omega_1)}$ and for any right ideal p of R let $G(p)$ denote the normal closure of $\{t(a,x) : a \in p,\ \text{unimodular } x \in R^{(\Omega_1)}\}$ in $G(R)$. Given $X \in Sp(\Omega,R)$ and $x,y \in R^{(\Omega_1)}$ we know that $(X(x),X(y)) = (x,y)$. Hence $t(a,x)^X = t(a,X^{-1}(x))$ and so we see that $G(R)$ and $G(p)$ are normal subgroups of $Sp(\Omega,R)$. Clearly $G(R)$ and $G(p)$ coincide with the 'elementary' subgroups of $Sp(\Omega,R)$ mentioned in Theorems 2.7 - 2.9. We shall now show that $G(R)$ and $G(p)$ coincide with $ESp(\Omega,R)$ and $ESp(\Omega,p)$ respectively.

Lemma 4.4. Let p be any right ideal of R; $ESp(\Omega,R) = G(R)$ and $ESp(\Omega,p) = G(p)$. In particular, $ESp(\Omega,R)$ and $ESp(\Omega,p)$ are normal subgroups of $Sp(\Omega,R)$.

Proof. Let $\{e_\lambda, e_{\lambda'} : \lambda \in \Omega\}$ be the canonical basis of $R^{(\Omega_1)}$. We see first that for $\lambda,\mu \in \Omega$, $\lambda \neq \mu$ and $a \in p$,

$$r_{\lambda\lambda}(a) = t(-a,e_\lambda),$$
$$r_{\lambda\mu}(a)r_{\lambda\lambda}(a)r_{\mu\mu}(a) = t(-a,e_\lambda + e_\mu),$$
$$s_{\lambda\lambda}(a) = t(a,e_{\lambda'}) \quad \text{and}$$

$$s_{\lambda\mu}(a)s_{\lambda\lambda}(a)s_{\mu\mu}(a) = t(a,e_{\lambda'} + e_{\mu'})$$

since $(e_\lambda,e_{\lambda'}) = 1 = -(e_{\lambda'},e_\lambda)$ and $(e_\lambda,e_{\mu'}) = 0 = (e_{\lambda'},e_\mu)$. Hence $ESp(\Omega,R) \leq G(R)$ and $ESp(\Omega,p) \leq G(p)$. It remains to prove the opposite inclusions. To do this, we first assert that whenever $x \in R^{(\Omega_1)}$ is unimodular there exist $\omega \in \Omega$ and $X \in ESp(\Omega,R)$ such that $X(e_\omega) = x$. We establish this assertion by translating the proof of Proposition 2.2 of [28] (which is only for commutative R but in fact holds without this assumption) in to terms of $r_{\lambda\mu}(a)$ and $s_{\lambda\mu}(a)$. Notice that for any $\lambda,\mu \in \Omega$ and $a \in p$

$$t(a,e_\lambda + e_{\mu'}) = t_{\lambda\mu}(a)s_{\mu\mu}(a)r_{\lambda\lambda}(-a)$$
$$t(a,e_\lambda - e_{\mu'}) = t_{\lambda\mu}(-a)s_{\mu\mu}(a)r_{\lambda\lambda}(-a)$$
$$t(a,-e_\lambda + e_{\mu'}) = t_{\lambda\mu}(-a)s_{\mu\mu}(a)r_{\lambda\lambda}(-a)$$

and

$$t(a,e_{\mu'} - e_\lambda + e_{\lambda'} - e_\mu) = (s_{\mu\lambda}(a)s_{\mu\mu}(a)s_{\lambda\lambda}(a))^{r_{\lambda\lambda}(1)r_{\mu\mu}(1)}.$$

Hence $t(a,e_\lambda + e_{\mu'})$, $t(a,e_\lambda - e_{\mu'})$, $t(a,-e_\lambda + e_{\mu'})$ and $t(a,e_{\mu'} - e_\lambda + e_{\lambda'} - e_\mu)$ all lie in $ESp(\Omega,p)$. If $x \in R^{(\Omega_1)}$ is unimodular, say $x = \sum_\lambda e_\lambda x_\lambda + e_{\lambda'}x_{\lambda'}$, then there exist $a_\lambda, a_{\lambda'} \in R$ such that $\sum_\lambda a_\lambda x_\lambda + a_{\lambda'}x_{\lambda'} = 1$. We shall suppose that there exists $\lambda \in \Omega$ such that $x_\lambda \neq 0$ (for if not then there exists $\lambda' \in \Omega'$ such that $x_{\lambda'} \neq 0$ and the resulting proof is similar to the one we shall give). For each $\alpha \neq \lambda$ define

$$A_\alpha = t(-x_\alpha - x_{\alpha'} - x_\alpha x_{\alpha'},e_{\lambda'})t(-x_{\alpha'},e_{\alpha'})t(x_{\alpha'},e_{\lambda'} + e_{\alpha'})t(x_\alpha,e_{\lambda'} + e_\alpha).$$

Pick $\mu \in \Omega$ such that $x_\mu = x_{\mu'} = 0$ and define

$$B = t(-x_\lambda + 1,e_{\mu'} - e_\lambda + e_{\lambda'} - e_\mu)t(x_\lambda - 1,e_{\mu'} - e_\lambda)t((x_\lambda - 1)(1 + x_{\lambda'}),$$
$$e_{\lambda'} - e_\mu)t(x_{\lambda'},e_{\lambda'}).$$

For each $\alpha \neq \mu$ define

$$C_\alpha = t((x_\lambda - 1)a_\alpha,e_{\alpha'} + e_{\mu'})t(-(x_\lambda - 1)a_\alpha,e_{\alpha'})t(-(x_\lambda - 1)a_{\alpha'},e_\alpha + e_{\mu'}).$$
$$t((x_\lambda - 1)a_{\alpha'},e_\alpha).$$

Put $A = \prod_{\alpha \neq \lambda} A_\alpha$ and $C = \prod_{\alpha \neq \mu} C_\alpha$. Then $A,B,C \in ESp(\Omega,R)$ and so if $D = CBA$, $D \in ESp(\Omega,R)$. Moreover $A(e_\lambda) = \sum_{\alpha \neq \lambda} e_\alpha x_\alpha + e_{\alpha'} x_{\alpha'} + e_\lambda$ and $BA(e_\lambda) = A(e_\lambda) - e_{\mu'}(x_\lambda - 1) + e_\lambda(x_\lambda - 1) + e_{\lambda'} x_{\lambda'}$. Thus $D(e_\lambda) = x$.

Now let $a \in R$ and let x be a unimodular element in $R^{(\Omega_1)}$. We shall show that $t(a,x) \in ESp(\Omega,R)$. If $x = \Sigma\, e_\lambda x_\lambda + e_{\lambda'} x_{\lambda'}$, we may suppose, as before, that $x_\lambda \neq 0$, for some $\lambda \in \Omega$. Then there exists $X \in ESp(\Omega,R)$ such that $x = X(e_\lambda)$ and we may deduce that

$$t(a,x) = t(a,X(e_\lambda)) = t(a,e_\lambda)^{X^{-1}} = r_{\lambda\lambda}(-a)^{X^{-1}} \qquad (*)$$

and that $t(a,x) \in ESp(\Omega,R)$. Further, if $a \in p$ then (*) shows that $t(a,x) \in ESp(\Omega,p)$ since $ESp(\Omega,p)$ is a normal subgroup of $ESp(\Omega,R)$. This completes the proof of the lemma.

In [28] Maxwell shows that if R is a commutative ring and p is an ideal of R

$$G(p) = [G(R),G(p)] = [G(R),Sp'(\Omega,p)].$$

In fact, the proof given in [28] does not require R to be commutative. This observation, together with the lemma, allow us to state

Corollary 4.1. Let p be a two sided ideal of R;

$$ESp(\Omega,p) = [ESp(\Omega,R),ESp(\Omega,p)] = [ESp(\Omega,R),Sp'(\Omega,p)].$$

Since the classification of the subnormal ~~subnormal~~ subgroups of $Sp(\Omega,R)$ will depend on $Sp'(\Omega,p)$, for ideals p of R, it will be useful to obtain a description of the centre of $Sp(\Omega,R)$. To do this we use

Lemma 4.5. If $X \in Sp(\Omega,R)$ commutes with $t_{\lambda\mu}(x)$ then

(i) $X_{\alpha\lambda} x = 0 = X_{\alpha\mu'} x$ for all $\alpha \in \Omega_1$, $\alpha \neq \lambda,\mu'$,

(ii) $xX_{\mu\beta} = 0 = xX_{\lambda'\beta}$ for all $\beta \in \Omega_1$, $\beta \neq \lambda', \mu$,

(iii) $X_{\lambda\mu'}x = -xX_{\mu\lambda'}$

(iv) $xX_{\lambda'\mu} = -X_{\mu'\lambda}x$

(v) $X_{\lambda\lambda}x = xX_{\mu\mu}$

(vi) $xX_{\lambda'\lambda'} = X_{\mu'\mu'}x$.

Proof. Let $t_{\lambda\mu}(x) = 1 + E$. Then X commutes with $t_{\lambda\mu}(x)$ if and only if X commutes with E. However

(i) for all $\beta \in \Omega_1$, $\beta \neq \lambda', \mu$ $(XE)_{\alpha\beta} = 0$ while $(XE)_{\alpha\mu} = X_{\alpha\lambda}x$ and $(XE)_{\alpha\lambda'} = -X_{\alpha\mu'}x$.

(ii) for all $\alpha \in \Omega_1$, $\alpha \neq \lambda, \mu'$ $(EX)_{\alpha\beta} = 0$ while $(EX)_{\lambda\beta} = xX_{\mu\beta}$ and $(EX)_{\mu'\beta} = -xX_{\lambda'\beta}$. In particular

(iii) $(XE)_{\lambda\mu} = X_{\lambda\lambda}x$, $(EX)_{\lambda\mu} = xX_{\mu\mu}$,

$(XE)_{\mu'\lambda'} = -X_{\mu'\mu'}x$, $(EX)_{\mu'\lambda'} = -xX_{\lambda'\lambda'}$,

(iv) $(XE)_{\lambda\lambda'} = -X_{\lambda\mu'}x$, $(EX)_{\lambda\lambda'} = xX_{\mu\lambda'}$,

$(XE)_{\mu'\mu} = X_{\mu'\lambda}x$, $(EX)_{\mu'\mu} = -xX_{\lambda'\mu}$.

Thus from (i) and (ii) we see that for all $\alpha \in \Omega_1$, $\alpha \neq \lambda, \mu'$, $X_{\alpha\lambda}x = 0$ and $X_{\alpha\mu'}x = 0$ and for all $\beta \neq \mu, \lambda'$, $xX_{\mu\beta} = 0 = xX_{\lambda'\beta}$. The other conclusions of the lemma are deduced from (iii) and (iv).

Corollary 4.2. The centre of $Sp(\Omega,R)$ comprises $\begin{bmatrix} rI & 0 \\ 0 & rI \end{bmatrix}$, for all central units r of R with $r^2 = 1$. The centre of $ESp(\Omega,R)$ is trivial.

Proof. Let $X \in Z(Sp(\Omega,R))$. X commutes with $t_{\lambda\mu}(1)$ for all $\lambda, \mu \in \Omega$, $\lambda \neq \mu$. From Lemma 4.5 we see that $X = \begin{bmatrix} rI & 0 \\ 0 & sI \end{bmatrix}$ where r, s

are units of R since X is invertible. X also commutes with $t_{\lambda\mu}(x)$ for all $x \in R$ and so, again from Lemma 4.5, we see that r and s are central. Moreover, X commutes with $r_{\lambda\lambda}(1)$ which shows that r = s. Finally, since X is symplectic, we see that $r^2 = 1$. It is clear that $\begin{bmatrix} rI & 0 \\ 0 & rI \end{bmatrix}$, where r is a central unit and $r^2 = 1$, is in the centre of $Sp(\Omega,R)$ and this proves the first part of the lemma. We see that the centre of $ESp(\Omega,R)$ is trivial since matrices in $ESp(\Omega,R)$ differ from the $\Omega_1 \times \Omega_1$ identity matrix in at most finitely many positions.

Lemma 4.6. If $X \in Sp(\Omega,R)$ and R is a commutative ring then J(X) is the least ideal p such that $X \in Sp'(\Omega,p)$.

Proof. We need to show that $X \in Sp'(\Omega,J(X))$ and that whenever $X \in Sp'(\Omega,p)$ then $J(X) \leq p$. Let $X \in Sp(\Omega,R)$. $X \equiv \begin{bmatrix} rI & 0 \\ 0 & rI \end{bmatrix} \bmod J(X)$ for some $r \in R$ such that $r^2 \in J(X)$ and rJ(X) is a unit of R/J(X). By Corollary 4.2 we see that $X \in Sp'(\Omega,J(X))$. Suppose that $X \in Sp'(\Omega,p)$; it follows that $X \equiv \begin{bmatrix} rI & 0 \\ 0 & rI \end{bmatrix} \bmod p$ and that $X_{\alpha\beta} \equiv 0 \bmod p$ and $X_{\alpha\alpha} \equiv X_{\beta\beta} \bmod p$, for all $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$. Hence $J(X) \leq p$.

Lemma 4.7. Let R be a commutative ring; $X \in Z(Sp(\Omega,R))$ if and only if J(X) = 0.

Proof. If $X \in Z(Sp(\Omega,R))$ then Corollary 4.2 shows that J(X) = 0. If J(X) = 0 then $X \in Sp'(\Omega,0) = Z(Sp(\Omega,R))$.

Lemma 4.8. Let R be a commutative ring. If H is a subgroup of $Sp(\Omega,R)$ that is normalized by $ESp(\Omega,R)$ then the following assertions are equivalent.

(i) $H \leq Z(Sp(\Omega,R))$,

(ii) $[H,ESp(\Omega,R)] = 1$,

(iii) $H \cap ESp(\Omega,R) = 1$.

Proof. If H is central and $X \in H \cap ESp(\Omega,R)$ then $X \in Z(ESp(\Omega,R)) = 1$, by Corollary 4.2 and so (i) implies (iii). Since H is normalized by $ESp(\Omega,R)$, $[H,ESp(\Omega,R)] \leq H \cap ESp(\Omega,R)$ and this shows that (iii) implies (ii). Finally, if $[H,ESp(\Omega,R)] = 1$ then, whenever $X \in H$, X commutes with $t_{\lambda\mu}(x)$ and $r_{\lambda\lambda}(1)$ for all $x \in R$ and all $\lambda,\mu \in \Omega$, $\lambda \neq \mu$. Thus, by Lemma 4.5 and Corollary 4.2 we see that $X \in Z(Sp(\Omega,R))$ and this shows that (ii) implies (i) and completes the proof of the lemma.

Definition 4.6. Let $H \leq Sp(\Omega,R)$. We define $K(H)$ to be the two sided ideal $\Sigma\, J(X)$, where the sum is taken over all those $X \in H \cap ESp(\Omega,R)$.

We see that whenever R is a commutative ring and H is a non-central subgroup of $Sp(\Omega,R)$ normalized by $ESp(\Omega,R)$ then $K(H) \neq 0$. For, otherwise $K(H) = 0$ and so $H \cap ESp(\Omega,R) = 1$ which, by Lemma 4.8, shows that H is central, contrary to hypothesis. We also remark that since matrices in $ESp(\Omega,R)$ differ from the $\Omega_1 \times \Omega_1$ identity matrix in at most finitely many positions, if $X \in H \cap ESp(\Omega,R)$ then $J(X)$ is generated by the matrix entries $X_{\alpha\beta}$ and $X_{\alpha\alpha} - 1$, for all $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$.

For $H \leq Sp(\Omega,R)$ we define a generator of $J(H)$ to be either $X_{\alpha\beta}$ or

$X_{\alpha\alpha} - X_{\beta\beta}$, for some $X \in H$ and $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$. For $X \in H \cap ESp(\Omega,R)$, we define a generator of $J(X)$ to be either $X_{\alpha\beta}$ or $X_{\alpha\alpha} - 1$, for some $\alpha,\beta \in \Omega_1$, $\alpha \neq \beta$.

Lemma 4.9. If H is a subgroup of $Sp(\Omega,R)$ that is normalized by $ESp(\Omega,p)$, for some two sided ideal p of R then $K(H)$ contains yx for all generators y of $J(H)$ and all $x \in p$.

Proof. Let $x \in p$. There are several cases to consider.

(a) $y = A_{\mu\rho}$, for some $\mu,\rho \in \Omega$, $\mu \neq \rho$ and $A \in H$. Pick $\lambda \in \Omega$, $\lambda \neq \rho,\mu$ and put $t = t_{\rho\lambda}(x)$. Then $[t,A^{-1}] \in ESp(\Omega,R) \cap H$, by hypothesis and so for all $\psi \in \Omega_1$, $\psi \neq \mu$, $[t,A^{-1}]_{\mu\psi} \in K(H)$ and $[t,A^{-1}]_{\mu\mu} - 1 \in K(H)$. But

$$[t,A^{-1}]_{\mu\psi} = A_{\mu\rho}xA^{-1}_{\lambda\psi} - A_{\mu\lambda'}xA^{-1}_{\rho'\psi}, \qquad \psi \neq \mu$$

and

$$[t,A^{-1}]_{\mu\mu} - 1 = A_{\mu\rho}xA^{-1}_{\lambda\mu} - A_{\mu\lambda'}xA^{-1}_{\rho'\mu}.$$

Thus

$$\begin{aligned}&([t,A^{-1}]_{\mu\mu} - 1)A_{\mu\lambda} + \sum_{\psi \neq \mu} [t,A^{-1}]_{\mu\psi}A_{\psi\lambda}\\ &= A_{\mu\rho}xA^{-1}_{\lambda\mu}A_{\mu\lambda} - A_{\mu\lambda'}xA^{-1}_{\rho'\mu}A_{\mu\lambda} + \sum_{\psi \neq \mu} A_{\mu\rho}xA^{-1}_{\lambda\psi}A_{\psi\lambda} - \sum_{\psi \neq \mu} A_{\mu\lambda'}xA^{-1}_{\rho'\psi}A_{\psi\lambda}\\ &= A_{\mu\rho}x\end{aligned}$$

and hence $A_{\mu\rho}x \in K(H)$.

(b) $y = A_{\mu'\rho}$, for some $\mu,\rho \in \Omega$, $A \in H$. Let λ and t be as in case (a). This time we note that $K(H)$ contains $[t,A^{-1}]_{\mu'\mu'} - 1$ and $[t,A^{-1}]_{\mu'\psi}$, for all $\psi \in \Omega_1$, $\psi \neq \mu'$. But

$$[t,A^{-1}]_{\mu'\psi} = A_{\mu'\rho}xA^{-1}_{\lambda\psi} - A_{\mu'\lambda'}xA^{-1}_{\rho'\psi}, \qquad \psi \neq \mu'$$

and

$$[t,A^{-1}]_{\mu'\mu'} - 1 = A_{\mu'\rho}xA^{-1}_{\lambda\mu'} - A_{\mu'\lambda'}xA^{-1}_{\rho'\mu'}.$$

Thus

$$([t,A^{-1}]_{\mu'\mu'} - 1)A_{\mu'\lambda} + \sum_{\psi\neq\mu'} [t,A^{-1}]_{\mu'\psi}A_{\psi\lambda}$$

$$= A_{\mu'\rho}xA^{-1}_{\lambda\mu'}A_{\mu'\lambda} - A_{\mu'\lambda'}xA^{-1}_{\rho'\mu'}A_{\mu'\lambda} + \sum_{\psi\neq\mu'} A_{\mu'\rho}xA^{-1}_{\lambda\psi}A_{\psi\lambda}$$

$$- \sum_{\psi\neq\mu'} A_{\mu'\lambda'}xA^{-1}_{\rho'\psi}A_{\psi\lambda}.$$

$$= A_{\mu'\rho}x.$$

and hence $A_{\mu'\rho}x \in K(H)$.

(c) $y = A_{\mu'\rho'}$, for some $\mu,\rho \in \Omega$, $\mu \neq \rho$, $A \in H$. Pick $\lambda \in \Omega$, $\lambda \neq \rho,\mu$ and put $t = t_{\lambda\rho}(x)$. Then $[t,A^{-1}] \in ESp(\Omega,R) \cap H$, by hypothesis and so $[t,A^{-1}]_{\mu'\mu'} - 1 \in K(H)$ and $[t,A^{-1}]_{\mu'\psi} \in K(H)$ for all $\psi \in \Omega_1$, $\psi \neq \mu'$. But

$$[t,A^{-1}]_{\mu'\psi} = A_{\mu'\lambda}xA^{-1}_{\rho\psi} - A_{\mu'\rho'}xA^{-1}_{\lambda'\psi} \qquad \psi \neq \mu'$$

and

$$[t,A^{-1}]_{\mu'\mu'} - 1 = A_{\mu'\lambda}xA^{-1}_{\rho\mu'} - A_{\mu'\rho'}xA^{-1}_{\lambda'\mu'}.$$

Thus

$$([t,A^{-1}]_{\mu'\mu'} - 1)A_{\mu'\lambda'} + \sum_{\psi\neq\mu'} [t,A^{-1}]_{\mu'\psi}A_{\psi\lambda'}$$

$$= A_{\mu'\lambda}xA^{-1}_{\rho\mu'}A_{\mu'\lambda'} - A_{\mu'\rho'}xA^{-1}_{\lambda'\mu'}A_{\mu'\lambda'} + \sum_{\psi\neq\mu'} A_{\mu'\lambda}xA^{-1}_{\rho\psi}A_{\psi\lambda'}$$

$$- \sum_{\psi\neq\mu'} A_{\mu'\rho'}xA^{-1}_{\lambda'\psi}A_{\psi\lambda'}$$

$$= -A_{\mu'\rho'}x$$

and hence $A_{\mu'\rho'}x \in K(H)$.

(d) $y = A_{\mu\rho'}$, for some $\mu,\rho \in \Omega$, $A \in H$. Let λ and t be as in case (c). This time we note that $K(H)$ contains $[t,A^{-1}]_{\mu\mu} - 1$ and $[t,A^{-1}]_{\mu\psi}$, for all $\psi \in \Omega_1$, $\psi \neq \mu$. But

$$[t,A^{-1}]_{\mu\psi} = A_{\mu\lambda}xA^{-1}_{\rho\psi} - A_{\mu\rho'}xA^{-1}_{\lambda'\psi}, \qquad \psi \neq \mu$$

and

$$[t,A^{-1}]_{\mu\mu} - 1 = A_{\mu\lambda}xA^{-1}_{\rho\mu} - A_{\mu\rho'}xA^{-1}_{\lambda'\mu}.$$

Thus

$$([t,A^{-1}]_{\mu\mu} - 1)A_{\mu\lambda'} + \sum_{\psi \neq \mu} [t,A^{-1}]_{\mu\psi}A_{\psi\lambda'}$$

$$= A_{\mu\lambda}xA^{-1}_{\rho\mu}A_{\mu\lambda'} - A_{\mu\rho'}xA^{-1}_{\lambda'\mu}A_{\mu\lambda'} + \sum_{\psi \neq \mu} A_{\mu\lambda}xA^{-1}_{\rho\psi}A_{\psi\lambda'}$$

$$- \sum_{\psi \neq \mu} A_{\mu\rho'}xA^{-1}_{\lambda'\psi}A_{\psi\lambda'}$$

$$= -A_{\mu\rho'}x$$

and hence $A_{\mu\rho'}x \in K(H)$.

(e) $y = A_{\mu\mu} - A_{\rho\rho}$, for some $\mu,\rho \in \Omega$, $\mu \neq \rho$, $A \in H$. Let λ and t be as in case (a). We see that $K(H)$ contains $[t,A^{-1}]_{\rho\rho} - 1$ and $[t,A^{-1}]_{\rho\psi}$ for all $\psi \in \Omega_1$, $\psi \neq \rho$. But

$$[t,A^{-1}]_{\rho\psi} = A_{\rho\rho}xA^{-1}_{\lambda\psi} - A_{\rho\lambda'}xA^{-1}_{\rho'\psi} - xA_{\lambda\rho}xA^{-1}_{\lambda\psi} + xA_{\lambda\lambda'}xA^{-1}_{\rho'\psi}, \quad \psi \neq \rho,\lambda$$

$$[t,A^{-1}]_{\rho\rho} - 1 = A_{\rho\rho}xA^{-1}_{\lambda\rho} - A_{\rho\lambda'}xA^{-1}_{\rho'\rho}$$

$$[t,A^{-1}]_{\rho\lambda} = A_{\rho\rho}xA^{-1}_{\lambda\lambda} - A_{\rho\lambda'}xA^{-1}_{\rho'\lambda} - x.$$

But $A_{\rho\lambda'}x \in K(H)$ from case (d) so that $K(H)$ contains $A_{\rho\rho}xA^{-1}_{\lambda\lambda} - x$, $A_{\rho\rho}xA^{-1}_{\lambda\rho}$ and $A_{\rho\rho}xA^{-1}_{\lambda\psi}$, for all $\psi \neq \rho,\lambda$. Thus $K(H)$ contains

$$\sum_{\psi \neq \rho,\lambda} A_{\rho\rho}xA^{-1}_{\lambda\psi}A_{\psi\lambda} + A_{\rho\rho}xA^{-1}_{\lambda\rho}A_{\rho\lambda} + A_{\rho\rho}xA^{-1}_{\lambda\lambda}A_{\lambda\lambda} - xA_{\lambda\lambda}$$

$$= A_{\rho\rho}x - xA_{\lambda\lambda}.$$

Similarly we see that $K(H)$ contains $A_{\mu\mu}x - xA_{\lambda\lambda}$ and hence $K(H)$ contains $(A_{\mu\mu} - A_{\rho\rho})x$.

(f) $y = A_{\mu'\mu'} - A_{\rho'\rho'}$, for some $\mu,\rho \in \Omega$, $\mu \neq \rho$, $A \in H$. Let λ and t be as in case (c). We see that $K(H)$ contains $[t,A^{-1}]_{\rho'\rho'} - 1$ and $[t,A^{-1}]_{\rho'\psi}$, for all $\psi \in \Omega_1$, $\psi \neq \rho'$. But

$$[t,A^{-1}]_{\rho'\psi} = A_{\rho'\lambda}xA^{-1}_{\rho\psi} - A_{\rho'\rho'}xA^{-1}_{\lambda'\psi} \qquad \psi \neq \rho',\lambda'$$

$$[t,A^{-1}]_{\rho'\rho'} - 1 = A_{\rho'\lambda}xA^{-1}_{\rho\rho'} - A_{\rho'\rho'}xA^{-1}_{\lambda'\rho'}$$

$$[t,A^{-1}]_{\rho'\lambda'} = A_{\rho'\lambda}xA^{-1}_{\rho\lambda'} - A_{\rho'\rho'}xA^{-1}_{\lambda'\lambda'} + x.$$

In a way similar to that employed in case (e) we see that $xA_{\lambda'\lambda'} - A_{\rho'\rho'}x$ lies in $K(H)$ and that $xA_{\lambda'\lambda'} - A_{\mu'\mu'}x \in K(H)$ and conclude that $K(H)$ contains $(A_{\mu'\mu'} - A_{\rho'\rho'})x$.

(g) $y = A_{\rho\rho} - A_{\mu'\mu'}$, for some $\mu, \rho \in \Omega$, $A \in H$. Pick $\lambda \in \Omega$, $\lambda \neq \rho, \mu$ and put $r = r_{\lambda\rho}(x)$. Then $[r, A^{-1}] \in ESp(\Omega, R) \cap H$, by hypothesis and so, for all $\psi \in \Omega_1$, $\psi \neq \rho, \lambda'$, $K(H)$ contains $[r, A^{-1}]_{\rho\psi}$ and $[r, A^{-1}]_{\rho\rho} - 1$. But

$$[r,A^{-1}]_{\rho\psi} = A_{\rho\rho} x A^{-1}_{\lambda'\psi} + A_{\rho\lambda} x A^{-1}_{\rho'\psi}, \qquad \psi \neq \rho, \lambda'$$

$$[r,A^{-1}]_{\rho\rho} - 1 = A_{\rho\rho} x A^{-1}_{\lambda'\rho} + A_{\rho\lambda} x A^{-1}_{\rho'\rho}$$

$$[r,A^{-1}]_{\rho\lambda'} = A_{\rho\rho} x A^{-1}_{\lambda'\lambda'} + A_{\rho\lambda} x A^{-1}_{\rho'\lambda'} - x.$$

Thus, as in cases (e) and (f) we see that $K(H)$ contains $A_{\rho\rho}x - xA_{\lambda'\lambda'}$. However, case (f) shows that $K(H)$ contains $A_{\mu'\mu'}x - xA_{\lambda'\lambda'}$ so that $K(H)$ contains $(A_{\rho\rho} - A_{\mu'\mu'})x$ also. This completes the proof of case (g) and of the lemma.

Corollary 4.3. If H is a subgroup of $Sp(\Omega, R)$ that is normalized by $ESp(\Omega, p)$, for some two sided ideal p of R then $J(H)p \leq K(H)$.

Proof. If $x \in J(H)p$ then $x = \sum_{i=1}^{n} s_i y_i r_i$, where the y_i are generators of $J(H)$ and $s_i \in R$, $r_i \in p$, $i = 1, \dots, n$. The lemma shows that each $y_i r_i \in K(H)$; thus $x \in K(H)$. We deduce that $J(H)p \leq K(H)$.

Lemma 4.10. Let R be a commutative ring and let H be a subgroup of $Sp(\Omega, R)$ that is normalized by $ESp(\Omega, p)$, for some ideal p of R. Let $A \in H \cap ESp(\Omega, R)$ and let x be any generator of $J(A)$. Let $u_i \in p$, $i = 1, \dots, 5$, let $y = 4u_5u_4u_3u_2u_1x$ and let $\lambda, \mu \in \Omega$, $\lambda \neq \mu$. H contains $r_{\lambda\mu}(y)$, $r_{\lambda\lambda}(y)$, $s_{\lambda\mu}(y)$ and $s_{\lambda\lambda}(y)$.

Proof. We may suppose that $x \neq 0$, for otherwise there is nothing to prove. There are several cases to consider.

(a) $x = A_{\alpha\beta}$, for some $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. Let $\varphi \in \Omega$, $\varphi \neq \lambda, \mu, \alpha, \beta$, be such that the φth column and the φ'th row of A are trivial. We first examine the effect of $[t_{\varphi\alpha}(u_1), A]$ on the canonical basis $\{e_\omega, e_{\omega'} : \omega \in \Omega\}$ of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[t_{\varphi\alpha}(u_1), A]e_\omega = e_\omega + e_\varphi u_1 A_{\alpha\omega}, \qquad \omega \neq \varphi, \alpha$$

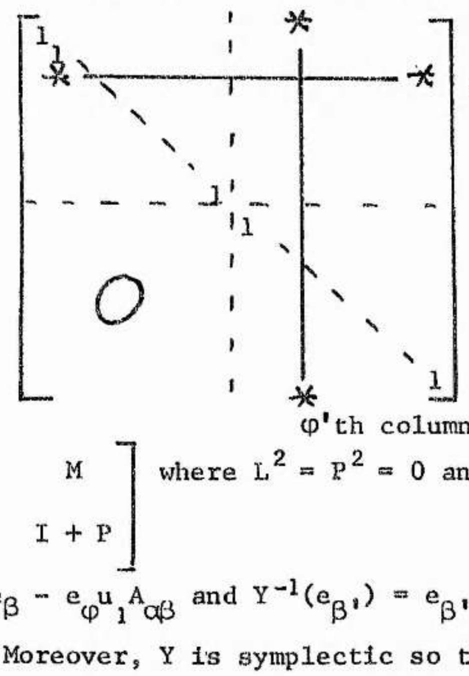
$$[t_{\varphi\alpha}(u_1), A]e_\alpha = e_\alpha + e_\varphi u_1 (A_{\alpha\alpha} - 1)$$

$$[t_{\varphi\alpha}(u_1), A]e_\varphi = e_\varphi$$

$$[t_{\varphi\alpha}(u_1), A]e_{\omega'} = e_{\omega'} + e_\varphi u_1 A_{\alpha\omega'}, \qquad \omega \neq \varphi$$

$$[t_{\varphi\alpha}(u_1), A]e_{\varphi'} = e_{\varphi'} + e_{\alpha'}u_1 - A^{-1}(e_{\alpha'})u_1 + e_\varphi u_1(A_{\alpha\varphi'} + A^{-1}_{\alpha\alpha'}u_1).$$

We see that $Y = [t_{\varphi\alpha}(u_1), A]$ differs from 1 in only the φth row and the φ'th column; in other words Y has the form



φth row

φ'th column

say $Y = \begin{bmatrix} I + L & M \\ 0 & I + P \end{bmatrix}$ where $L^2 = P^2 = 0$ and $Y^{-1} = \begin{bmatrix} I - L & -(I-L)M(I-P) \\ 0 & I-P \end{bmatrix}$.

Thus $Y^{-1}(e_\beta) = e_\beta - e_\varphi u_1 A_{\alpha\beta}$ and $Y^{-1}(e_{\beta'}) = e_{\beta'} - e_\varphi u_1 A_{\alpha\beta'}$. Hence $Y^{-1}_{\varphi\beta} = -u_1 A_{\alpha\beta}$. Moreover, Y is symplectic so that $(I+P)' = I-L$ and $Y_{\beta'\varphi'} = Y^{-1}_{\varphi\beta}$, that is $Y_{\beta'\varphi'} = -u_1 A_{\alpha\beta}$. We now examine the effect of $[r_{\varphi\beta}(-u_2), Y]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[r_{\varphi\beta}(-u_2), Y]e_\omega = e_\omega$$

$$[r_{\varphi\beta}(-u_2), Y]e_{\omega'} = e_{\omega'}, \qquad \omega \neq \varphi$$

$$[r_{\varphi\beta}(-u_2), Y]e_{\varphi'} = e_{\varphi'} - e_\varphi(u_2 Y_{\beta'\varphi'} - u_2 u_1 A_{\alpha\beta})$$

$$= e_{\varphi'} + e_\varphi 2u_2 u_1 A_{\alpha\beta}.$$

It follows that whenever $\varphi \in \Omega$, $\varphi \neq \lambda,\mu,\alpha,\beta$ is such that the φth column and the φ'th row of A are trivial, H contains $r_{\varphi\varphi}(2u_2u_1x)$. Let $\psi \in \Omega$, $\psi \neq \varphi,\lambda,\mu,\alpha,\beta$ be such that the ψ'th row of A and the ψth column of A are trivial. We see that

$$[r_{\varphi\varphi}(z),t_{\psi\varphi}(u)] = r_{\psi\varphi}(-uz)r_{\psi\psi}(u^2z).$$

Hence H contains $r_{\psi\varphi}(2u_3u_2u_1x)$ because H contains $r_{\psi\psi}(2u_3^2u_2u_1x)$ since the choice of u_1, u_2 above was arbitrary. Finally, the identities

$$r_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [r_{\psi\varphi}(2u_3u_2u_1x),t_{\mu\varphi}(u_4),t_{\lambda\psi}(2u_5)]$$
$$r_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [r_{\psi\varphi}(2u_3u_2u_1x),t_{\lambda\varphi}(u_4),t_{\lambda\psi}(u_5)],$$
$$s_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [s_{\varphi\lambda}(2u_5),[s_{\psi\mu}(-u_4),r_{\psi\varphi}(2u_3u_2u_1x)]]$$
$$s_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [s_{\varphi\lambda}(u_5),[s_{\psi\lambda}(-u_4),r_{\psi\varphi}(2u_3u_2u_1x)]]$$

show that H contains the required matrices and completes the proof of case (a).

(b) $x = A_{\alpha\alpha} - 1$, for some $\alpha \in \Omega$, Let φ and Y be as in case (a). We see that $Y_{\alpha'\varphi'} = Y^{-1}_{\varphi\alpha} = -u_1(A_{\alpha\alpha} - 1)$. If we examine the effect of the commutator $[r_{\varphi\alpha}(-u_2),Y]$ on the canonical basis of $R^{(\Omega_1)}$ we find that

$$[r_{\varphi\alpha}(-u_2),Y]e_\omega = e_\omega \qquad \text{for all } \omega \in \Omega,$$
$$[r_{\varphi\alpha}(-u_2),Y]e_{\omega'} = e_{\omega'} \qquad \text{for all } \omega \in \Omega,\ \omega \neq \varphi$$
$$[r_{\varphi\alpha}(-u_2),Y]e_{\varphi'} = e_{\varphi'} + e_\varphi 2u_2u_1(A_{\alpha\alpha} - 1).$$

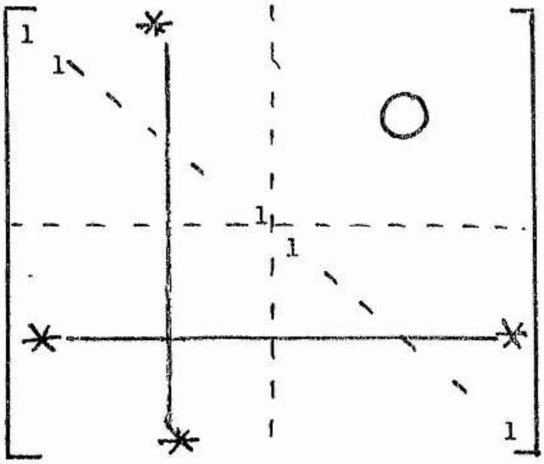
Hence H contains $r_{\varphi\varphi}(2u_2u_1x)$ and the proof of case (b) is completed in exactly the same way as case (a).

(c) $x = A_{\alpha'\beta'}$, for some $\alpha,\beta \in \Omega$, $\alpha \neq \beta$. Let $\varphi \in \Omega$, $\varphi \neq \lambda,\mu,\alpha,\beta$ be such that the φth row and the φ'th column of A are trivial. We first examine the effect of $[t_{\alpha\varphi}(u_1),A]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[t_{\alpha\varphi}(u_1),A]e_\omega = e_\omega - e_{\varphi'}u_1A_{\alpha'\omega}, \qquad \omega \neq \varphi$$
$$[t_{\alpha\varphi}(u_1),A]e_\varphi = e_\varphi - e_\alpha u_1 + A^{-1}(e_\alpha)u_1 + e_{\varphi'}u_1(A^{-1}_{\alpha'\alpha}u_1 - A_{\alpha'\varphi})$$
$$[t_{\alpha\varphi}(u_1),A]e_{\omega'} = e_{\omega'} - e_{\varphi'}u_1A_{\alpha'\omega'}, \qquad \omega \neq \varphi,\alpha$$
$$[t_{\alpha\varphi}(u_1),A]e_{\alpha'} = e_{\alpha'} - e_{\varphi'}u_1(A_{\alpha'\alpha'} - 1)$$

$$[t_{\alpha\varphi}(u_1),A]\, e_{\varphi'} = e_{\varphi'}.$$

We see that $Y = [t_{\alpha\varphi}(u_1),A]$ differs from 1 in only the φ'th row and the φth column. In other words, Y has the form



φ'th row

φth column

say, $Y = \begin{bmatrix} I+L & 0 \\ M & I+P \end{bmatrix}$ where $L^2 = P^2 = 0$ and $Y^{-1} = \begin{bmatrix} I-L & 0 \\ -(I-P)M(I-L) & I-P \end{bmatrix}$.

Thus $Y^{-1}(e_\beta) = e_\beta + e_{\varphi'}u_1A_{\alpha'\beta}$ and $Y^{-1}(e_{\beta'}) = e_{\beta'} + e_{\varphi'}u_1A_{\alpha'\beta'}$. Hence $Y^{-1}_{\varphi'\beta'} = u_1A_{\alpha'\beta'}$. Moreover, Y is symplectic so that $I+P = (I-L)'$ and $Y_{\beta\varphi} = Y^{-1}_{\varphi'\beta'}$, that is $Y_{\beta\varphi} = u_1A_{\alpha'\beta'}$. We now examine the effect of $[s_{\varphi\beta}(u_2),Y]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[s_{\varphi\beta}(u_2),Y]e_{\omega'} = e_{\omega'} \qquad \text{for all } \omega \in \Omega,$$

$$[s_{\varphi\beta}(u_2),Y]e_{\omega} = e_{\omega} \qquad \omega \neq \varphi,$$

$$\begin{aligned}[s_{\varphi\beta}(u_2),Y]e_{\varphi} &= e_\varphi + e_{\varphi'}(u_2u_1A_{\alpha'\beta'} + u_2Y_{\beta\varphi}) \\ &= e_\varphi + e_{\varphi'}2u_2u_1A_{\alpha'\beta'}.\end{aligned}$$

It follows that whenever $\varphi \in \Omega$, $\varphi \neq \lambda,\mu,\alpha,\beta$ is such that the φth row and the φ'th column of A are trivial, H contains $s_{\varphi\varphi}(2u_2u_1x)$. Let $\psi \in \Omega$, $\psi \neq \varphi,\lambda,\mu,\alpha,\beta$ be such that the ψth row and the ψ'th column of A are trivial. We see that

$$[s_{\varphi\varphi}(z),t_{\varphi\psi}(u)] = s_{\varphi\psi}(zu)s_{\psi\psi}(u^2z).$$

Hence H contains $s_{\varphi\psi}(2u_3u_2u_1x)$ because H contains $s_{\psi\psi}(2u^2u_3u_2u_1x)$ since the choice of u_1, u_2 above was arbitrary. Finally, the identities

$$s_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [s_{\varphi\psi}(2u_3u_2u_1x),t_{\varphi\mu}(u_4),t_{\psi\lambda}(2u_5)]$$
$$s_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [s_{\varphi\psi}(2u_3u_2u_1x),t_{\varphi\lambda}(u_4),t_{\psi\lambda}(u_5)]$$
$$r_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [r_{\varphi\lambda}(2u_5),[r_{\psi\mu}(-u_4),s_{\varphi\psi}(2u_3u_2u_1x)]]$$
$$r_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [r_{\lambda\varphi}(u_5),[r_{\psi\lambda}(-u_4),s_{\varphi\psi}(2u_3u_2u_1x)]]$$

show that H contains the required matrices and complete the proof of case (c).

(d) $x = A_{\alpha'\alpha'} - 1$, for some $\alpha \in \Omega$. Let φ and Y be as in case (c). We see that $Y^{-1}_{\varphi'\alpha'} = Y_{\alpha\varphi} = u_1(A_{\alpha'\alpha'} - 1)$. If we examine the effect of the commutator $[s_{\varphi\alpha}(u_2),Y]$ on the canonical basis of $R^{(\Omega_1)}$ we find that

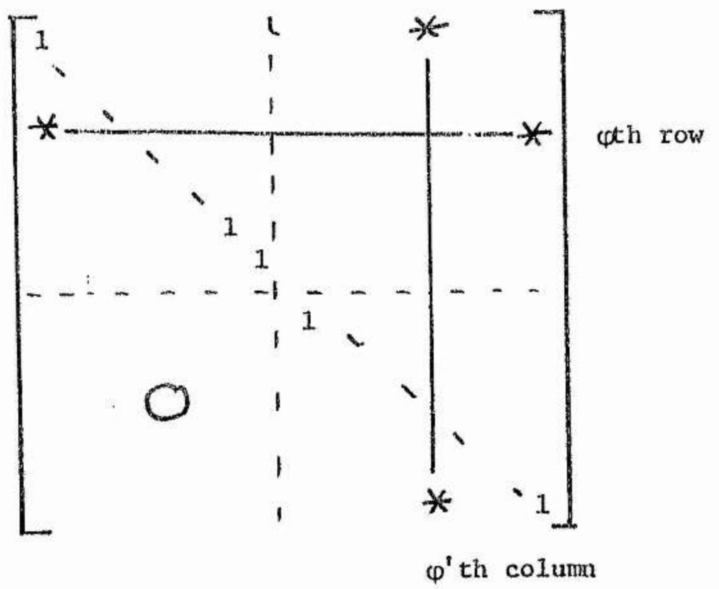
$$[s_{\varphi\alpha}(u_2),Y]\, e_{\omega'} = e_{\omega'} \qquad \text{for all } \omega \in \Omega,$$
$$[s_{\varphi\alpha}(u_2),Y]\, e_{\omega} = e_{\omega} \qquad \text{for all } \omega \in \Omega,\ \omega \neq \varphi,$$
$$[s_{\varphi\alpha}(u_2),Y]\, e_{\varphi} = e_{\varphi} + e_{\varphi'}2u_2u_1(A_{\alpha'\alpha'} - 1).$$

Hence H contains $s_{\varphi\varphi}(2u_2u_1x)$ and the proof of case (d) is completed in exactly the same way as case (c).

(e) $x = A_{\alpha'\beta'}$, for some $\alpha,\beta \in \Omega$. Let $\varphi \in \Omega$, $\varphi \neq \lambda,\mu,\alpha,\beta$ be such that the φth column of A and the φ'th row of A are trivial. We first examine the effect of the commutator$[r_{\varphi\alpha}(u_1),A]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[r_{\varphi\alpha}(u_1),A]e_{\omega} = e_{\omega} + e_{\varphi}u_1A_{\alpha'\omega}, \qquad \omega \neq \varphi$$
$$[r_{\varphi\alpha}(u_1),A]e_{\varphi} = e_{\varphi},$$
$$[r_{\varphi\alpha}(u_1),A]e_{\omega'} = e_{\omega'} + e_{\varphi}u_1A_{\alpha'\omega'}, \qquad \omega \neq \varphi,\alpha,$$
$$[r_{\varphi\alpha}(u_1),A]e_{\alpha'} = e_{\alpha'} + e_{\varphi}u_1(A_{\alpha'\alpha'} - 1)$$
$$[r_{\varphi\alpha}(u_1),A]e_{\varphi'} = e_{\varphi'} + e_{\varphi}u_1(A_{\alpha'\varphi'} - A^{-1}_{\alpha'\alpha}u_1) - e_{\alpha}u_1 + A^{-1}(e_{\alpha})u_1.$$

We see that $Y = [r_{\varphi\alpha}(u_1),A]$ differs from 1 in only the φth row and the φ'th column, in other words, Y has the form



say, $Y = \begin{bmatrix} I+L & M \\ 0 & I+P \end{bmatrix}$ and $Y^{-1} = \begin{bmatrix} I-L & -(I-L)M(I-P) \\ 0 & I-P \end{bmatrix}$

Thus $Y^{-1}(e_\beta) = e_\beta - e_\varphi u_1 A_{\alpha'\beta}$ and $Y^{-1}(e_{\beta'}) = e_{\beta'} - e_\varphi u_1 A_{\alpha'\beta'}$. However Y is symplectic so that $(I+P)' = I-L$ and $Y_{\beta'\varphi'} = Y^{-1}_{\varphi\beta} = -u_1 A_{\alpha'\beta}$. We now examine the effect of $[r_{\varphi\beta}(-u_2), Y]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[r_{\varphi\beta}(-u_2),Y]e_\omega = e_\omega$$

$$[r_{\varphi\beta}(-u_2),Y]e_{\omega'} = e_{\omega'} \qquad \omega \neq \varphi$$

$$[r_{\varphi\beta}(-u_2),Y]e_{\varphi'} = e_{\varphi'} + e_\varphi u_2 u_1 (A_{\alpha'\beta} \mp Y_{\beta'\varphi'})$$

$$= e_{\varphi'} + e_\varphi 2u_2 u_1 A_{\alpha'\beta}.$$

Hence H contains $r_{\varphi\varphi}(2u_2u_1x)$ and the proof of case (e) is completed in exactly the same way as case (a).

(f) $x = A_{\alpha\beta'}$; for some $\alpha,\beta \in \Omega$. Let $\varphi \in \Omega$, $\varphi \neq \lambda,\mu,\alpha,\beta$ be such that the φth row and the φ'th column of A are trivial. We first examine the effect of the commutator $[s_{\varphi\alpha}(u_1),A]$ on the canonical basis of $R^{(\Omega_1)}$. Let $\omega \in \Omega$.

$$[s_{\varphi\alpha}(u_1),A]e_\omega = e_\omega + e_{\varphi'} u_1 A_{\alpha\omega}, \qquad \omega \neq \varphi, \alpha$$

$$[s_{\varphi\alpha}(u_1),A]e_\alpha = e_\alpha + e_{\varphi'} u_1 (A_{\alpha\alpha} - 1)$$

$$[s_{\varphi\alpha}(u_1),A]e_{\varphi} = e_{\varphi} - e_{\alpha'}u_1 + A^{-1}(e_{\alpha'})u_1 + e_{\varphi'}u_1(A_{\alpha\varphi} - A^{-1}_{\varphi\alpha}).$$

$$[s_{\varphi\alpha}(u_1),A]e_{\omega'} = e_{\omega'} + e_{\varphi'}u_1A_{\alpha\omega'}, \qquad \omega \neq \varphi$$

$$[s_{\varphi\alpha}(u_1),A]e_{\varphi'} = e_{\varphi'}.$$

We see that $Y = [s_{\varphi\alpha}(u_1),A]$ differs from 1 in only the φ'th row and the φth column. As before we see that $Y_{\beta\varphi} = Y^{-1}_{\varphi'\beta'} = -u_1A_{\alpha\beta'}$, since Y is symplectic. If we now examine the effect of $[s_{\varphi\beta}(u_2),Y]$ on the canonical basis of $R^{(\Omega_1)}$ we find that

$$[s_{\varphi\beta}(-u_2),Y]e_{\omega'} = e_{\omega'} \qquad \text{for all } \omega \in \Omega,$$

$$[s_{\varphi\beta}(-u_2),Y]e_{\omega} = e_{\omega} \qquad \text{for all } \omega \in \Omega,\ \omega \neq \varphi,$$

$$[s_{\varphi\beta}(-u_2),Y]e_{\varphi} = e_{\varphi} + e_{\varphi'}u_2(u_1A_{\alpha\beta'} - Y_{\beta\varphi})$$

$$= e_{\varphi} + e_{\varphi'}2u_2u_1A_{\alpha\beta'}.$$

Hence H contains $s_{\varphi\varphi}(2u_2u_1x)$ and the proof of case (f) is completed in exactly the same way as case (c). This also completes the proof of the lemma.

Corollary 4.4. If R is a commutative ring and if H is a subgroup of $Sp(\Omega,R)$ that is normalized by $ESp(\Omega,p)$, for some ideal p of R, then H contains $ESp(\Omega,4p^6J(H))$.

Proof. If H is normalized by $ESp(\Omega,p)$ then so also is H^Y, for any $Y \in ESp(\Omega,R)$. Moreover, $J(H) = J(H^Y)$ by Lemma 4.1. We see from the lemma that H contains $\{r_{\lambda\mu}(y),\ s_{\lambda\mu}(y)\ :\ \lambda,\mu \in \Omega,\ y \in 4p^5K(H)\}$, since if $y \in 4p^5K(H)$, $y = \Sigma\, 4u_{5i}u_{4i}u_{3i}u_{2i}u_{1i}x_i$, where the x_i are generators of $K(X_i)$, for some $X_i \in H$ and the $u_{ji} \in p$. However. Corollary 4.3 shows that $J(H)p \leq K(H)$ and so we deduce that H contains $S = \{r_{\lambda\mu}(y),\ s_{\lambda\mu}(y): \lambda,\mu \in \Omega,\ y \in 4p^6J(H)\}$. Our opening remarks show that S is also contained in H^Y, for all $Y \in ESp(\Omega,R)$ so that

$S \le \mathrm{Core}_{ESp(\Omega,R)} H = H_0$, say. Hence H_0 contains the normal closure of S in $ESp(\Omega,R)$ which is just $ESp(\Omega, 4p^6 J(H))$ and, therefore, so also does H as required.

We now come to the main theorem of this chapter. It gives a classification of the normal and subnormal subgroups of $Sp(\Omega,R)$ along the lines of Theorem 3.1 and Theorem 2.2. It also generalizes the main result of [28].

Theorem 4.1. Let R be a commutative ring. If G is a subgroup of $Sp(\Omega,R)$ containing $ESp(\Omega,R)$ and if H is a subnormal subgroup of G, say

$$H = H_d \triangleleft H_{d-1} \triangleleft \dots \triangleleft H_1 \triangleleft H_0 = G$$

then there exists an ideal p of R such that

$$ESp(\Omega, (4p)^{f(d)}) \le H \le Sp'(\Omega,p)$$

where $f(d) = (6^d - 1)/5$.

Proof. Whenever $p \le q$, $Sp'(\Omega,p) \le Sp'(\Omega,q)$. Thus, since $X \in Sp'(\Omega,J(X))$, from Lemma 4.6, we see that $X \in Sp'(\Omega,J(H))$, for all $X \in H$, and hence $H \le Sp'(\Omega,J(H))$.

To prove the first inclusion we shall use an inductive argument. If $d = 1$ then H is normalized by $ESp(\Omega,R)$ so that H contains $ESp(\Omega,4R^6J(H)) = ESp(\Omega,4J(H))$. Now suppose that $ESp(\Omega,(4J(K))^{f(k)}) \le K$ for all subgroups K with subnormal chains of length $k < d$. If we put $J_0 = J(H_{d-1})$ then we see that H_{d-1} contains $ESp(\Omega,(4J_0)^{f(d-1)})$. But $J = J(H) \le J_0$ so that H_{d-1} contains $ESp(\Omega,(4J)^{f(d-1)})$; thus $ESp(\Omega,(4J)^{f(d-1)})$ normalizes H. Therefore H contains $ESp(\Omega,4(4J)^{6f(d-1)}J) = ESp(\Omega,(4J)^{f(d)})$.

We conclude, by the principle of induction, that whenever H is as in the statement of the theorem $ESp(\Omega,(4p)^{f(d)}) \leq H$ when $p = J(H)$. This, together with our initial remarks, completes the proof of the theorem.

We now state, for the sake of completeness, what is essentially the main theorem of [28].

Corollary 4.5. Let R be a commutative ring and suppose that 2 is a unit of R. The following assertions are equivalent.

(i) H is a subgroup of $Sp(\Omega,R)$ normalized by $ESp(\Omega,R)$,

(ii) there exists a unique ideal p of R such that

$$ESp(\Omega,p) \leq H \leq Sp'(\Omega,p).$$

Proof. That (i) implies (ii) follows from Theorem 4.1, by using the fact that 2 is a unit. If we now suppose that (ii) holds then we see that

$$[ESp(\Omega,R),H] \leq [ESp(\Omega,R),Sp'(\Omega,p)] \leq ESp(\Omega,p) \leq H$$

by Corollary 4.1. That p is unique follows from the observation that $p = J(H)$ is minimal with respect to $H \leq Sp'(\Omega,p)$ and maximal with respect to $ESp(\Omega,p) \leq H$.

Corollary 4.5 shows that when R is a commutative ring in which 2 is a unit normal subgroups of $Sp(\Omega,R)$ are sandwiched in a way analogous to the sandwiching of normal subgroups of $GL(\Omega,R)$.

Next suppose that for some subgroup H of $ESp(\Omega,R)$, $ESp(\Omega,4J(H)) \leq H \leq Sp'(\Omega,J(H))$. In order to deduce that H is a normal subgroup of $ESp(\Omega,R)$, in general, we require that 2 is a unit of R, as our next example shows.

Example 4.1. Let $\Omega = N$ and $R = Z$. Let H be the subgroup of $ESp(N,Z)$ generated by $ESp(N,4Z)$ and $\{r_{ij}(a) : i,j \in N,\ a \in Z\}$. We see that $J(H) = Z$ and that $ESp(N,4Z) \leq H \leq Sp'(N,Z)$. However H is not a normal subgroup of $ESp(N,Z)$ since $t_{12}(1) \notin H$ but $t_{12}(1) = [r_{13}(1), s_{23}(1)]$ and this commutator belongs to the normal closure of H in $ESp(N,Z)$.

We include the next lemma and corollary for the sake of completeness, although they are not applied anywhere in the thesis.

Lemma 4.11. Let R be a commutative ring, let p be an ideal of R and let k be an integer , $k \geq 1$. $[Sp'(\Omega,p), Sp'(\Omega,p^k)] \leq Sp'(\Omega,p^{k+1})$.

Proof. Let $X \in Sp'(\Omega,p)$ and $Y \in Sp'(\Omega,p^k)$. Then by Corollary 4.2 and Corollary 3.2 we see that $X \in GL'(\Omega_1,p)$ and $Y \in GL'(\Omega_1,p^k)$. By the proof of Lemma 3.20 we see that $[X,Y] \equiv 1 \bmod p^{k+1}$ so that $[X,Y] \in Sp'(\Omega,p^{k+1})$ and this completes the proof of the lemma.

Corollary 4.6. Let R be a commutative ring and let p be an ideal of R. For any integer $k \geq 1$, $Sp'(\Omega,p)/Sp'(\Omega,p^k)$ is nilpotent of class at most k-1.

Proof. We assert that $\gamma_k(Sp'(\Omega,p)) \leq Sp'(\Omega,p^k)$. This is certainly true when $k = 1$. Thus suppose that $\gamma_{k-1}(Sp'(\Omega,p)) \leq Sp'(\Omega,p^{k-1})$. Then

$$\begin{aligned} \gamma_k(Sp'(\Omega,p)) &= [Sp'(\Omega,p), \gamma_{k-1}(Sp'(\Omega,p))] \\ &\leq [Sp'(\Omega,p), Sp'(\Omega,p^{k-1})] \\ &\leq Sp'(\Omega,p^k) \end{aligned}$$

by the lemma. Hence $\gamma_k(Sp'(\Omega,p)/Sp'(\Omega,p^k)) = 1$ and the corollary is proven.

Lemma 4.12. If R is a commutative ring in which 2 is a unit, if p is an ideal of R and if k is an integer, $k \geq 1$ then

$$ESp(\Omega,p^{r}) \leq \gamma_{r}(ESp(\Omega,p)).$$

Proof. From the proof of Corollary 4.6 we see that $\gamma_r(ESp(\Omega,p))$ is contained in $Sp'(\Omega,p^r)$ and so $\gamma_r(ESp(\Omega,p))$ has level p^r. An application of Corollary 4.5 now shows that $ESp(\Omega,p^r)$ is contained in $\gamma_r(ESp(\Omega,p))$.

Lemma 4.13. If R is a commutative ring, if p is an ideal of R and if k is an integer, $k \geq 1$, then

$$\gamma_{k+1}(Sp'(\Omega,p) \cap ESp(\Omega,R)) \leq ESp(\Omega,p^{k}).$$

Proof. We first note that $\gamma_{k+1}(Sp'(\Omega,p) \cap ESp(\Omega,R))$ is contained in $Sp'(\Omega,p^{k+1})$; this follows from the proof of Corollary4.6. Hence

$$\begin{aligned}\gamma_{k+1}(Sp'(\Omega,p) \cap ESp(\Omega,R)) &\leq [Sp'(\Omega,p^{k}),ESp(\Omega,R)]\\ &\leq ESp(\Omega,p^{k})\end{aligned}$$

and this completes the proof of the lemma.

Corollary 4.7. Whenever R is a commutative ring and k is an integer, $k \geq 1$, $(Sp'(\Omega,p) \cap ESp(\Omega,R))/ESp(\Omega,p^k)$ (and hence also $ESp(\Omega,p)/ESp(\Omega,p^k)$) is nilpotent of class at most k.

Proof. For any integer $r \geq 1$

$$\gamma_{r}((Sp'(\Omega,p) \cap ESp(\Omega,R))/ESp(\Omega,p^{k})) \leq \frac{\gamma_{r}(Sp'(\Omega,p) \cap ESp(\Omega,R))ESp(\Omega,p^{k})}{ESp(\Omega,p^{k})}$$

Thus the lemma shows that $\gamma_{k+1}(Sp'(\Omega,p) \cap ESp(\Omega,R)) = 1$.

Proposition 4.1. Let R be a commutative ring in which 2 is a unit. The following assertions are equivalent, for $H \leq ESp(\Omega, R)$.

(i) H is a subnormal subgroup of $ESp(\Omega,R)$,

(ii) for some ideal p of R and some integer m

$$ESp(\Omega,p^m) \leq H \leq Sp'(\Omega,p).$$

If (i) holds then the least m in (ii) satisfies $d-1 \leq m \leq f(d)$. If (ii) holds then the defect of H in $ESp(\Omega,R)$ is at most $m+1$. (f is as defined in Theorem 4.1.)

Proof. If (i) holds then (ii) follows from Theorem 4.1. Suppose that (ii) holds. Since $ESp(\Omega,p)/ESp(\Omega,p^m)$ is nilpotent there is a series

$$ESp(\Omega,p^m) \leq \gamma_m \leq \dots \leq \gamma_1 \leq ESp(\Omega,R)$$

where $\gamma_i = \gamma_i(ESp(\Omega,p))$, $i = 1,\dots,m$. But

$$[ESp(\Omega,R),\gamma_1 H] \leq ESp(\Omega,p)[ESp(\Omega,R),Sp'(\Omega,p)] \leq ESp(\Omega,p)$$

shows that

$$H \leq \gamma_m H \leq \dots \leq \gamma_1 H \leq ESp(\Omega,R)$$

is a normal series from H to $ESp(\Omega,R)$ of length $m+1$ and we conclude that H is a subnormal subgroup of $ESp(\Omega,R)$ of defect at most $m+1$.

It remains to show that if (i) holds then the least m in (ii) satifies $d-1 \leq m \leq f(d)$. It is clear that $m \leq f(d)$ since we can always take $m = f(d)$. The proof of (ii) implies (i) shows that, if (ii) holds then $d \leq m+1$, i.e. $d-1 \leq m$. This completes the proof of the proposition.

Proposition 4.1 shows that subgroups of $ESp(\Omega,R)$ are subnormal if and only if they are sandwiched by $ESp(\Omega,p^m)$ and $Sp'(\Omega,p)$ for some ideal p and some integer m. It also shows that $d-1 \leq m \leq f(d)$, where d is the defect of the subnormal subgroup. Clearly, for large d, there is a large discrepancy between d-1 and f(d) and we have not been able to find an example to show that it is necessary to take $m = f(d)$. However, our next example shows that we can, on occasion, take $m = d$.

Example 4.2. Let R, S_i, $S_{i,j}$ be as in Example 3.7. Then S_i is a subideal of defect i. For each i, denote by $T\langle S_i\rangle$ the set $\{r_{\lambda\mu}(x),\ s_{\lambda\mu}(x) : \lambda, \mu \in \Omega,\ x \in S_i\}$ and define, for $i \geq 2$, $ESp\langle S_i\rangle$ to be the normal closure of $T\langle S_i\rangle$ in $ESp\langle S_{i-1}\rangle$ where $ESp\langle S_1\rangle = ESp(\Omega, S_1)$. These definitions give rise to the chain

$$ESp\langle S_i\rangle \lhd \dots \lhd ESp\langle S_1\rangle \lhd ESp(\Omega,R)$$

of subnormal subgroups of $ESp(\Omega,R)$. It is clear that, for each $i \geq 1$ and $g \in Sp(\Omega,R)$, $J(ESp\langle S_i\rangle^g) = J(ESp\langle S_i\rangle) = S_1$ and that $S_1^i \leq S_i$. Hence $T\langle S_1^i\rangle$ is contained in $\mathrm{Core}_{ESp(\Omega,R)} ESp\langle S_i\rangle$, so we see that $ESp(\Omega, S^i) \leq ESp\langle S_i\rangle$. As in Example 3.8we see that $ESp\langle S_i\rangle$ has defect i in $ESp(\Omega,R)$, thus if we put $H = ESp\langle S_i\rangle$, we see that H is a subnormal subgroup of $ESp(\Omega,R)$ of defect i with $ESp(\Omega, J(H)^i) \leq H \leq Sp'(\Omega, J(H))$.

We now investigate the simplicity of $ESp(\Omega,R)$.

Proposition 4.2. Let R be a simple commutative ring with char $R \neq 2, 4$; $ESp(\Omega,R)$ is a simple group.

Proof. If H is a non-trivial normal subgroup of $ESp(\Omega,R)$ then $J(H) \neq 0$ and since char $R \neq 2, 4$, $0 \neq 4J(H) = R$. Hence H contains $ESp(\Omega,R)$ and we conclude that $ESp(\Omega,R)$ is simple.

Proposition 4.3. If R is a division ring, char $R \neq 2$, then $ESp(\Omega,R)$ is a simple group. If R is a division ring and H is a normal subgroup of $Sp(\Omega,R)$ then either H is central or H contains $ESp(\Omega,R)$.

Proof. Let H be a non-trivial normal subgroup of $ESp(\Omega,R)$; then, from Corollary 4.3, $K(H) = J(H) = R$. Moreover, from the proof of Lemma 4.10, case (a), we see that H contains $r_{\varphi\rho}(1)$ by taking $u_1 = (A_{\alpha\beta})^{-1}/2$ and $u_2 = 1$. Since the relations in Lemma 4.2 hold if one of a, b is 1 we see that H contains $r_{\lambda\mu}(x)$ and $s_{\lambda\mu}(x)$, for all $\lambda,\mu \in \Omega$ and all $x \in R$, from which it follows that H contains $ESp(\Omega,R)$ and hence $ESp(\Omega,R)$ is simple.

Now let H be a normal subgroup of $ESp(\Omega,R)$ and suppose that H does not contain $ESp(\Omega,R)$; then $[H,ESp(\Omega,R)] = 1$ and Lemma 4.5 and Corollary 4.2 show that $H \leq Z(Sp(\Omega,R))$.

In fact Proposition 4.3 is just a special case of Theorem 2.3 and Corollary 2.4 of [37]. We end chapter four with some miscellaneous results concerning subnormal subgroups of $ESp(\Omega,R)$.

Proposition 4.4. Let R be a commutative ring in which 2 is a unit.

(i) If $H_i \triangleleft^{d_i} ESp(\Omega,R)$, for $i = 1,\dots,r$ then $\langle H_1,\dots,H_r\rangle \triangleleft^{d} ESp(\Omega,R)$ where $d = \sum_{i=1}^{r} f(d_i) + 1$.

(ii) If the nil radical of every homomorphic image of R is nilpotent (for example, if R is Noetherian) and if $\{H_\alpha : \alpha \in A\}$ is a family of subnormal subgroups of $ESp(\Omega,R)$ then $\langle H_\alpha : \alpha \in A\rangle$ is also a subnormal subgroup of $ESp(\Omega,R)$.

Proof. (i) Let J_i denote the level of H_i and n_i denote $f(d_i)$, $i = 1,\dots,r$. By Theorem 4.1, each H_i contains $ESp(\Omega, J_i^{n_i})$. But $J(H_i) = J(H_i^{ESp(\Omega,R)})$ and $(Sp'(\Omega,J_i) \cap ESp(\Omega,R))/ESp(\Omega,J_i^{n_i})$ is nilpotent of class at most n_i and so, since

$$H_i^{ESp(\Omega,R)} \leq Sp'(\Omega,J_i) \cap ESp(\Omega,R)$$

(again by Theorem 4.1) it follows that

$$\gamma_{n_i+1}(H_i^{ESp(\Omega,R)}) \leq ESp(\Omega,J_i^{n_i}) \leq H_i.$$

Thus, by Theorem 1.1 we see that $\langle H_1,\dots,H_r\rangle$ is subnormal of defect at most $\sum_{i=1}^{r} f(d_i) + 1$.

(ii) For each $\alpha \in A$ write $J_\alpha = J(H_\alpha)$ and $n_\alpha = f(d_\alpha)$ where $H_\alpha \lhd^{d_\alpha} ESp(\Omega,R)$. Let J be the sum of the ideals J_α, for all $\alpha \in A$ and K the sum of all the ideals $J_\alpha^{n_\alpha}$, for all $\alpha \in A$. Then J/K is a nil ideal of R/K so that $J^r \leq K$, for some integer r. Therefore

$$\begin{aligned} ESp(\Omega,J^r) \leq ESp(\Omega,K) &\leq \langle ESp(\Omega,J^{n_\alpha}) : \alpha \in A\rangle \\ &\leq \langle H_\alpha : \alpha \in A\rangle \leq Sp'(\Omega,J). \end{aligned}$$

We conclude that $\langle H_\alpha : \alpha \in A\rangle$ is subnormal in $ESp(\Omega,R)$ by applying Proposition 4.2.

Chapter five

In chapter three and chapter four we have investigated the normal and subnormal subgroup structure of the infinite dimensional classical groups $GL(\Omega,R)$ and $Sp(\Omega,R)$. A key rôle in this classification was played by the 'elementary' subgroups $E(\Omega,R)$ and $ESp(\Omega,R)$ generated by the elementary matrices $t(\Lambda,f,\mu)$ and $r_{\lambda\mu}(x)$ and $s_{\lambda\mu}(x)$ respectively. We now restrict our attention to the group $E(\Omega,R)$ and we shall obtain a presentation for $E(\Omega,R)$ in terms of the $t(\Lambda,f,\mu)$, when R is a division ring. Much work of this nature has been done in the finite dimensional case; this was summarized in chapter two. Throughout this chapter, R shall denote a ring with identity, R^* shall denote the group of units of R and Ω shall denote an infinite set.

Let Λ be a proper subset of Ω and let $\mu \in \Omega-\Lambda$. Let $f:\Lambda \to R$; we shall extend f to be a mapping $f:\Omega \to R$ by defining $f(\omega) = 0$, for all $\omega \in \Omega-\Lambda$. With this choice of f, Λ and μ we shall call $s(\Lambda,f,\mu)$ a Steinberg symbol. We shall use Steinberg symbols to obtain presentations for $E(\Omega,R)$ and $EF(\Omega,R)$, when R is a division ring. Our first step in this direction is to define the Steinberg groups $St(\Omega,R)$ and $StF(\Omega,R)$. Let $\Lambda_1,\Lambda_2 \subset \Omega$, $\mu \in \Omega-(\Lambda_1 \cup \Lambda_2)$, $f:\Lambda_1 \to R$, $g:\Lambda_2 \to R$. We define the relation

S1: $s(\Lambda_1,f,\mu)s(\Lambda_2,g,\mu) = s(\Lambda_1 \cup \Lambda_2,f+g,\mu)$.

Let $\Lambda_1,\Lambda_2 \subset \Omega$, $\mu \in \Omega-\Lambda_1$, $\rho \in \Omega-\Lambda_2$, $f:\Lambda_1 \to R$ and $g:\Lambda_2 \to R$. We define the relation

S2:
$$[s(\Lambda_1,f,\mu),s(\Lambda_2,g,\rho)] = \begin{cases} s(\Lambda_2,f(\rho)g,\mu) & \mu \notin \Lambda_2 \\ s(\Lambda_1,-g(\mu)f,\rho) & \rho \notin \Lambda_1. \end{cases}$$

We shall denote by $St(\Omega,R)$ the group generated by all the Steinberg symbols $s(\Lambda,f,\mu)$, where $\Lambda \subset \Omega$, $\mu \in \Omega-\Lambda$ and $f:\Lambda \to R$, subject to the relations S1 and S2. When Λ is a finite set we shall call $s(\Lambda,f,\mu)$ a finite Steinberg symbol and we shall denote by $StF(\Omega,R)$ the group generated by all the finite Steinberg symbols subject to the relations S1 and S2. If $\Lambda = \{\lambda\}$ and $f(\lambda) = x$ we shall write $s(\Lambda,f,\mu)$ as $s(\lambda,x,\mu)$. When $\Omega = N$, $StF(\Omega,R)$ is the group $St(R)$ of [30] which has received much attention.

The assignment $\varphi: s(\Lambda,f,\mu) \to t(\Lambda,f,\mu)$ defines an epimorphism $\varphi: St(\Omega,R) \to E(\Omega,R)$. Similarly we obtain the epimorphism $\varphi_f: StF(\Omega,R) \to EF(\Omega,R)$ given by the assignment $s(\lambda,x,\mu) \to t(\lambda,x,\mu)$.

Definition 5.1. (i) $K_2(\Omega,R) = \ker \varphi$,

(ii) $KF_2(\Omega,R) = \ker \varphi_f$.

We see that $KF_2(N,R)$ is the well known abelian group $K_2(R)$.

Lemma 5.1. If R is a simple ring then $St(\Omega,R)/K_2(\Omega,R)$ and $StF(\Omega,R)/KF_2(\Omega,R)$ are simple groups.

Proof. It is immediate from definition 5.1 that $St(\Omega,R)/K_2(\Omega,R)$ is isomorphic to $E(\Omega,R)$ and that $StF(\Omega,R)/KF_2(\Omega,R) \approx EF(\Omega,R)$. The lemma follows from the work of chapter three.

In order to arrive at a presentation for $E(\Omega,R)$ we shall have to make several calculations with the Steinberg symbols. To facilitate these calculations, we make the following definitions.

Definition 5.2. Let $\lambda,\mu \in \Omega$, $\lambda \neq \mu$ and let x, y, $z \in R^*$. Define the symbols $w_{\lambda\mu}(x)$, $h_{\lambda\mu}(x)$ and $\{y,z\}_{\lambda\mu}$ by

$$\begin{aligned} w_{\lambda\mu}(x) &= s(\mu,x,\lambda)s(\lambda,-x^{-1},\mu)s(\mu,x,\lambda), \\ h_{\lambda\mu}(x) &= w_{\lambda\mu}(x)w_{\lambda\mu}(-1), \\ \{y,z\}_{\lambda\mu} &= h_{\lambda\mu}(y)h_{\lambda\mu}(z)h_{\lambda\mu}(zy)^{-1}. \end{aligned}$$

Notice that $w_{\lambda\mu}(x)^{-1} = w_{\lambda\mu}(-x)$ and $h_{\lambda\mu}(1) = 1$ since $s(\lambda, x,\mu)^{-1} = s(\lambda, -x,\mu)$ and $s(\lambda,0,\mu) = 1$, by relation S1.

If we let $f_{\lambda\mu}(x)$, $d_{\lambda\mu}(x)$ and $c(y,z)_{\lambda\mu}$ denote the images under φ of $w_{\lambda\mu}(x)$, $h_{\lambda\mu}(x)$ and $\{y,z\}_{\lambda\mu}$ respectively we see that

(i) $f_{\lambda\mu}(x)$ is the $\Omega \times \Omega$ matrix F, say, that differs from the $\Omega \times \Omega$ identity matrix in only the λth and μth rows with $F_{\lambda\alpha} = 0$ for all $\alpha \neq \mu$, $F_{\lambda\mu} = x$ and $F_{\mu\alpha} = 0$ for all $\alpha \neq \lambda$, $F_{\mu\lambda} = -x^{-1}$, that is

$$f_{\lambda\mu}(x) = \begin{array}{cl} \left[\begin{array}{ccccc} \ddots & \vdots & & \vdots & \\ & 1 & & \vdots & \\ \cdots & 0 & \cdots & x & \cdots \\ & \vdots & 1 & \vdots & \\ & \vdots & \ddots & \vdots & \\ & \vdots & & 1 & \\ \cdots & -x^{-1} & \cdots & 0 & \cdots \\ & \vdots & & 1 & \\ & \vdots & & \vdots & \ddots \end{array}\right] & \begin{array}{l} \\ \\ \lambda\text{th row} \\ \\ \\ \\ \mu\text{th row} \\ \\ \\ \end{array} \\ \begin{array}{ccc} \lambda\text{th column} & \quad & \mu\text{th column} \end{array} & \end{array}$$

(ii) $d_{\lambda\mu}(x)$ is the $\Omega \times \Omega$ diagonal matrix $\mathrm{diag}[\ldots,1,x,1,\ldots 1,x^{-1},1,\ldots]$ where x occurs in the λth place and x^{-1} in the μth place, that is

$$d_{\lambda\mu}(x) = \begin{array}{@{}c@{\quad}l@{}} \left[\begin{array}{ccccccccc} \ddots & & & & & & & & \\ & 1 & & & & & \text{\Large O} & & \\ & & x & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & x^{-1} & & \\ & & \text{\Large O} & & & & & 1 & \\ & & & & & & & & \ddots \end{array}\right] & \begin{array}{l} \\ \\ \lambda\text{th row} \\ \\ \\ \\ \mu\text{th row} \\ \\ \\ \end{array} \end{array}$$

(iii) $c(y,z)_{\lambda\mu}$ is the $\Omega \times \Omega$ diagonal matrix $\mathrm{diag}[\ldots,1,[y^{-1},z^{-1}],1,\ldots]$ with the commutator $[y^{-1},z^{-1}]$ occuring in the λth place, that is

$$c(y,z)_{\lambda\mu} = \begin{array}{@{}c@{\quad}l@{}} \left[\begin{array}{ccccccc} \ddots & & & & & & \\ & 1 & & & & \text{\Large O} & \\ & & 1 & & & & \\ & & & [y^{-1},z^{-1}] & & & \\ & & & & 1 & & \\ & \text{\Large O} & & & & 1 & \\ & & & & & & \ddots \end{array}\right] & \begin{array}{l} \\ \\ \\ \lambda\text{th row} \\ \\ \\ \\ \end{array} \end{array}$$

Our next few results indicate how the Steinberg symbols $w_{\alpha\beta}(x)$ and $h_{\alpha\beta}(x)$ interact. These results are very similar to the analogous results in Milnor [30].

Lemma 5.2. Conjugation by any $w_{\alpha\beta}(x)$ carries any Steinberg symbol into another Steinberg symbol and any finite Steinberg symbol into another finite Steinberg symbol. In particular, if $s(\Lambda,f,\mu)$ is a Steinberg symbol then $s(\Lambda,f,\mu)^{w_{\alpha\beta}(x)}$ is the Steinberg symbol:

(i) if $\alpha,\beta \notin \Lambda$, $\alpha,\beta \neq \mu$, $s(\Lambda,f,\mu)$,

(ii) if $\alpha,\beta \notin \Lambda$, $\alpha \neq \mu$, $\beta = \mu$, $s(\Lambda,-xf,\alpha)$,

(iii) if $\alpha,\beta \notin \Lambda$, $\alpha = \mu$, $\beta \neq \mu$, $s(\Lambda,x^{-1}f,\beta)$,

(iv) if $\alpha \notin \Lambda$, $\beta \in \Lambda$, $\alpha,\beta \neq \mu$, $s(\Lambda',f',\mu)$ where

$\Lambda' = \Lambda \cup \{\alpha\}-\{\beta\}$, $f'|_{\Lambda-\{\beta\}} = f$ and $f'(\alpha) = -f(\beta)x^{-1}$,

(v) if $\alpha \in \Lambda$, $\beta \notin \Lambda$, $\alpha,\beta \neq \mu$, $s(\Lambda',f',\mu)$ where

$\Lambda' = \Lambda \cup \{\beta\}-\{\alpha\}$, $f'|_{\Lambda-\{\alpha\}} = f$ and $f'(\beta) = f(\alpha)x$,

(vi) if $\alpha,\beta \in \Lambda$, $\alpha,\beta \neq \mu$, $s(\Lambda,f',\mu)$ where

$f'(\alpha) = -f(\beta)x^{-1}$, $f'(\beta) = f(\alpha)x$ and $f(\lambda) = f'(\lambda)$, for all $\lambda \neq \alpha,\beta$,

(vii) if $\alpha \in \Lambda$, $\beta \notin \Lambda$, $\alpha \neq \mu$, $\beta = \mu$, $s(\Lambda',-xf',\alpha)$ where

$\Lambda' = \Lambda \cup \{\beta\}-\{\alpha\}$, $f'|_{\Lambda-\{\alpha\}} = f$ and $f'(\beta) = f(\alpha)x$,

(viii) if $\alpha \notin \Lambda$, $\beta \in \Lambda$, $\alpha = \mu$, $\beta \neq \mu$, $s(\Lambda',x^{-1}f',\beta)$ where

$\Lambda' = \Lambda \cup \{\alpha\}-\{\beta\}$, $f'|_{\Lambda-\{\beta\}} = f$ and $f'(\alpha) = -f(\beta)x^{-1}$.

Proof. We shall not give the proof of all eight cases - only a representative selection. From relations S1 and S2 it is clear that $w_{\alpha\beta}(x)$ will affect only those Steinberg symbols $s(\Lambda,f,\mu)$ for which α or $\beta \in \Lambda$ or $\mu = \alpha$ or β. Thus (i) is immediate.

(ii) Here we assume that $\alpha,\beta \notin \Lambda$, $\alpha \neq \mu$, $\beta = \mu$. We shall split the proof into three stages. First notice that

$$\begin{aligned} s(\mu,-x,\alpha)s(\Lambda,f,\mu)s(\mu,x,\alpha) &= s(\Lambda,f,\mu)[s(\Lambda,f,\mu),s(\mu,x,\alpha)] \\ &= s(\Lambda,f,\mu)s(\Lambda,-xf,\alpha) \end{aligned}$$

by relation S2. Next observe that

$$\begin{aligned} &s(\alpha,x^{-1},\mu)s(\Lambda,f,\mu)s(\Lambda,-xf,\alpha)s(\alpha,-x^{-1},\mu) \\ &= s(\Lambda,f,\mu)s(\Lambda,-xf,\alpha)[s(\Lambda,-xf,\alpha),s(\alpha,-x^{-1},\mu)] \\ &= s(\Lambda,f,\mu)s(\Lambda,-xf,\alpha)s(\Lambda,-f,\mu) = s(\Lambda,-xf,\alpha) \end{aligned}$$

by relation S1 and S2. We finally see that

$$s(\mu,-x,\alpha)s(\Lambda,-xf,\alpha)s(\mu,x,\alpha) = s(\Lambda,-xf,\alpha)$$

by relation S1 and this establishes (ii). The proof of (iii) is similar.

(iv) Here we suppose that $\alpha \notin \Lambda$, $\beta \in \Lambda$, $\alpha, \beta \neq \mu$. Again we shall establish this result in three stages. First notice that

$$s(\beta,-x,\alpha)s(\Lambda,f,\mu)s(\beta,x,\alpha) = s(\Lambda,f,\mu)$$

by relation S2. Next observe that

$$s(\alpha,x^{-1},\beta)s(\Lambda,f,\mu)s(\alpha,-x^{-1},\beta) = s(\Lambda,f,\mu)[s(\Lambda,f,\mu),s(\alpha,-x^{-1},\beta)]$$
$$= s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)$$

by relation S2. . We finally see that

$$s(\beta,-x,\alpha)s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)s(\beta,x,\alpha)$$
$$= s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)[s(\alpha,-f(\beta)x^{-1},\mu),s(\beta,x,\alpha)]$$
$$= s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)s(\beta,-f(\beta),\mu)$$
$$= s(\Lambda',f',\mu)$$

where Λ' and f' are as in the statement of the lemma. The proof of (v) is similar.

(vi) Here we assume that $\alpha,\beta \in \Lambda$, $\alpha,\beta \neq \mu$. We see that

$$s(\beta,-x,\alpha)s(\Lambda,f,\mu)s(\beta,x,\alpha) = s(\Lambda,f,\mu)[s(\Lambda,f,\mu),s(\beta,x,\alpha)]$$
$$= s(\Lambda,f,\mu)s(\beta,f(\alpha)x,\mu)$$

and that

$$s(\alpha,x^{-1},\beta)s(\Lambda,f,\mu)s(\beta,f(\alpha)x,\mu)s(\alpha,-x^{-1},\beta)$$
$$= s(\Lambda,f,\mu)s(\alpha,x^{-1},\beta)[s(\alpha,x^{-1},\beta),s(\Lambda,f,\mu)]s(\beta,f(\alpha)x,\mu)s(\alpha,-x^{-1},\beta)$$
$$= s(\Lambda,f,\mu)s(\alpha,x^{-1},\beta)s(\alpha,-f(\beta)x^{-1},\mu)s(\beta,f(\alpha)x,\mu)s(\alpha,-x^{-1},\beta)$$
$$= s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)s(\beta,f(\alpha)x,\mu)[s(\beta,f(\alpha)x,\mu),s(\alpha,-x^{-1},\beta)]$$
$$= s(\Lambda,f,\mu)s(\alpha,-f(\beta)x^{-1},\mu)s(\beta,f(\alpha)x,\mu)s(\alpha,-f(\alpha),\mu)$$
$$= s(\Lambda'',f'',\mu)$$

where $\Lambda'' = \Lambda$, $f''(\alpha) = -f(\beta)x^{-1}$, $f''(\beta) = f(\beta) + f(\alpha)x$ and $f''(\lambda) = f(\lambda)$ for all $\lambda \neq \alpha,\beta$. Finally we see that

$$
\begin{aligned}
&s(\beta,-x,\alpha)s(\Lambda'',f'',\mu)s(\beta,x,\alpha)\\
&= s(\Lambda'',f'',\mu)[s(\Lambda'',f'',\mu),s(\beta,x,\alpha)]\\
&= s(\Lambda'',f'',\mu)s(\beta,-f(\beta),\mu)\\
&= s(\Lambda',f',\mu)
\end{aligned}
$$

where Λ' and f' are as in the statement of the lemma.

(vii) Here we assume that $\alpha \in \Lambda$, $\beta \notin \Lambda$, $\alpha \neq \mu$ and $\beta = \mu$ and we again subdivide the proof into several parts.

First notice that $s(\Lambda,f,\mu) = s(\alpha,f(\alpha),\beta)s(\Lambda'',f'',\beta)$ where $\Lambda'' = \Lambda-\{\alpha\}$ and $f'' = f|_{\Lambda''}$. Then we see that

$$s(\Lambda,f,\beta)^{w_{\alpha\beta}(x)} = s(\alpha,f(\alpha),\beta)^{w_{\alpha\beta}(x)} s(\Lambda'',f'',\beta)^{w_{\alpha\beta}(x)}$$

However

$$s(\Lambda'',f'',\beta)^{w_{\alpha\beta}(x)} = s(\Lambda'',-xf'',\alpha)$$

by (ii) above. Moreover, for $\lambda \in \Omega$, $\lambda \neq \alpha,\beta$

$$
\begin{aligned}
s(\alpha,f(\alpha),\beta)^{w_{\alpha\beta}(x)} &= [s(\lambda,f(\alpha),\beta)^{w_{\alpha\beta}(x)},s(\alpha,1,\lambda)^{w_{\alpha\beta}(x)}]\\
&=[s(\lambda,-xf(\alpha),\alpha),s(\beta,x,\lambda)]\\
&= s(\beta,-xf(\alpha)x,\alpha).
\end{aligned}
$$

Thus $s(\Lambda,f,\mu)^{w_{\alpha\beta}(x)} = s(\Lambda',-xf',\alpha)$ where Λ' and f' are as in the statement of the lemma. The proof of (viii) is similar.

This lemma shows how the Steinberg symbols are affected by conjugation by the $w_{\alpha\beta}(x)$. We shall also find it useful to know how the $w_{\alpha\beta}(x)$ are themselves moved by similar conjugations.

Corollary 5.1. Conjugation by any $w_{\alpha\beta}(x)$ carries any $w_{\lambda\mu}(y)$ into some $w_{\varphi\rho}(z)$ for $\alpha,\beta,\lambda,\mu,\varphi,\rho \in \Omega$ and $x,y,z \in R^*$. In particular

$w_{\lambda\mu}(y)^{w_{\alpha\beta}(x)}$ is the symbol

(i) $w_{\lambda\mu}(y)$ if λ, μ are distinct from α, β,

(ii) $w_{\beta\mu}(x^{-1}y)$ if μ, β are distinct, $\alpha = \lambda$

(iii) $w_{\lambda\alpha}(-yx^{-1})$ if λ, α are distinct, $\mu = \beta$

(iv) $w_{\alpha\mu}(-xy)$ if α, μ are distinct, $\lambda = \beta$

(v) $w_{\lambda\beta}(yx)$ if λ, β are distinct, $\alpha = \mu$

(vi) $w_{\beta\alpha}(-x^{-1}yx^{-1})$ if $\lambda = \alpha$ and $\mu = \beta$,

(vii) $w_{\alpha\beta}(-xyx)$ if $\lambda = \beta$ and $\mu = \alpha$.

Proof. (i) follows immediately from (i) of the lemma.

(ii)
$$\begin{aligned} w_{\alpha\mu}(y)^{w_{\alpha\beta}(x)} &= s(\mu,y,\alpha)^{w_{\alpha\beta}(x)}\, s(\alpha,-y^{-1},\mu)^{w_{\alpha\beta}(x)}\, s(\mu,y,\alpha)^{w_{\alpha\beta}(x)} \\ &= s(\mu,x^{-1}y,\beta)\, s(\beta,-y^{-1}x,\mu)\, s(\mu,x^{-1}y,\beta) \\ &= w_{\beta\mu}(x^{-1}y) \end{aligned}$$

(iii)
$$\begin{aligned} w_{\lambda\beta}(y)^{w_{\alpha\beta}(x)} &= s(\beta,y,\lambda)^{w_{\alpha\beta}(x)}\, s(\lambda,-y^{-1},\beta)^{w_{\alpha\beta}(x)}\, s(\beta,y,\lambda)^{w_{\alpha\beta}(x)} \\ &= s(\alpha,-yx^{-1},\lambda)\, s(\lambda,xy^{-1},\alpha)\, s(\alpha,-yx^{-1},\lambda) \\ &= w_{\lambda\alpha}(-yx^{-1}) \end{aligned}$$

(iv)
$$\begin{aligned} w_{\beta\mu}(y)^{w_{\alpha\beta}(x)} &= s(\mu,y,\beta)^{w_{\alpha\beta}(x)}\, s(\beta,-y^{-1},\mu)^{w_{\alpha\beta}(x)}\, s(\mu,y,\beta)^{w_{\alpha\beta}(x)} \\ &= s(\mu,-xy,\alpha)\, s(\alpha,y^{-1}x^{-1},\mu)\, s(\mu,-xy,\alpha) \\ &= w_{\alpha\mu}(-xy) \end{aligned}$$

(v)
$$\begin{aligned} w_{\lambda\alpha}(y)^{w_{\alpha\beta}(x)} &= s(\alpha,y,\lambda)^{w_{\alpha\beta}(x)}\, s(\lambda,-y^{-1},\alpha)^{w_{\alpha\beta}(x)}\, s(\alpha,y,\lambda)^{w_{\alpha\beta}(x)} \\ &= s(\beta,yx,\lambda)\, s(\lambda,-x^{-1}y^{-1},\beta)\, s(\beta,yx,\lambda) \\ &= w_{\lambda\beta}(yx) \end{aligned}$$

(vi)
$$\begin{aligned} w_{\alpha\beta}(y)^{w_{\alpha\beta}(x)} &= s(\beta,y,\alpha)^{w_{\alpha\beta}(x)}\, s(\alpha,-y^{-1},\beta)^{w_{\alpha\beta}(x)}\, s(\beta,y,\alpha)^{w_{\alpha\beta}(x)} \\ &= s(\alpha,-x^{-1}yx^{-1},\beta)\, s(\beta,xy^{-1}x,\alpha)\, s(\alpha,-x^{-1}yx^{-1},\beta) \\ &= w_{\beta\alpha}(-x^{-1}yx^{-1}) \end{aligned}$$

$$\text{(vii)} \qquad w_{\beta\alpha}(y)^{w_{\alpha\beta}(x)} = s(\alpha,y,\beta)^{w_{\alpha\beta}(x)} s(\beta,-y^{-1},\alpha)^{w_{\alpha\beta}(x)} s(\alpha,y,\beta)^{w_{\alpha\beta}(x)}$$

$$= s(\beta,-xyx,\alpha)s(\alpha,x^{-1}y^{-1}x^{-1},\beta)s(\beta,-xyx,\alpha)$$

$$= w_{\alpha\beta}(-xyx)$$

and this case completes the proof of the corollary.

Many of the calculations in the Steinberg group will involve manipulating not only the $w_{\alpha\beta}(x)$ but also the $h_{\lambda\mu}(x)$. For this reason we include the next lemma.

Lemma 5.3. Let $x \in R^*$ and $\lambda,\mu,\varphi \in \Omega$, λ,μ,φ all distinct.

(i) $h_{\lambda\mu}(x)h_{\mu\lambda}(x) = 1$,

(ii) $h_{\lambda\mu}(x)^{-1}h_{\mu\varphi}(x)^{-1}h_{\varphi\lambda}(x)^{-1} = 1$.

Proof. We shall first examine the product

$$h_{\varphi\mu}(x)w_{\lambda\mu}(1)h_{\varphi\mu}(x)^{-1}.$$

It is equal to

$$w_{\lambda\mu}(1)^{w_{\varphi\mu}(1)w_{\varphi\mu}(-x)}$$

by the definition of the $h_{\varphi\mu}(x)$, and by Corollary 5.1 we see that this expression is just $w_{\lambda\mu}(x)$. Thus

$$h_{\varphi\mu}(x)w_{\lambda\mu}(1)h_{\varphi\mu}(x)^{-1}w_{\lambda\mu}(-1) = h_{\lambda\mu}(x).$$

However, if we examine the product

$$w_{\lambda\mu}(1)h_{\varphi\mu}(x)^{-1}w_{\lambda\mu}(-1)$$

we see that it is equal to

$$w_{\varphi\mu}(1)^{w_{\lambda\mu}(-1)} w_{\varphi\mu}(-x)^{w_{\lambda\mu}(-1)}$$

which is just $w_{\varphi\lambda}(1)w_{\varphi\lambda}(-x) = h_{\varphi\lambda}(x)^{-1}$. Thus

$$h_{\lambda\mu}(x) = h_{\varphi\mu}(x)h_{\varphi\lambda}(x)^{-1}.$$

But this also shows that

$$h_{\mu\lambda}(x) = h_{\varphi\lambda}(x)h_{\varphi\mu}(x)^{-1}$$

and hence $h_{\lambda\mu}(x)h_{\mu\lambda}(x) = 1$. (This also shows that $h_{\lambda\mu}(x)^{-1} = h_{\mu\lambda}(x)$ and that for any $\lambda, \mu, \varphi \in \Omega$, λ, μ, φ all distinct, $h_{\lambda\mu}(x)$ can be written in terms of $h_{\varphi\lambda}(x)$ and $h_{\varphi\mu}(x)$.)

To establish (ii) we note from (i) that

$$h_{\lambda\mu}(x) = h_{\mu\varphi}(x)^{-1}h_{\varphi\lambda}(x)^{-1}$$

so that

$$h_{\lambda\mu}(x)^{-1}h_{\mu\varphi}(x)^{-1}h_{\varphi\lambda}(x)^{-1} = 1$$

and this completes the proof of the lemma.

In order to obtain a presentation for $E(\Omega,R)$ we shall express elements of $\mathrm{St}(\Omega,R)$ in a factorized form. Some of the factors will be 'upper triangular' in the sense given by our next two definitions.

Definition 5.3. Let $<$ be a well ordering of Ω. We define $M(\Omega,R,<)$ to be the subgroup of $St(\Omega,R)$ generated by all the Steinberg symbols $s(\Lambda,f,\mu)$ for which $\Lambda \subset \Omega$, $\mu \in \Omega-\Lambda$, $f:\Lambda \to R$ and $\mu < \Lambda$. (By $\mu < \Lambda$ we mean that $\mu < \lambda$, for all $\lambda \in \Lambda$.)

Definition 5.4. Let $<$ be a well ordering of Ω. We define $MF(\Omega,R,<)$ to be the subgroup of $StF(\Omega,R)$ generated by all the finite Steinberg symbols $s(\lambda,x,\mu)$ for which $\lambda,\mu \in \Omega$, $\mu < \lambda$ and $x \in R$.

We may regard these symbols as 'upper triangular' since their images under φ are upper triangular $\Omega \times \Omega$ matrices. We shall need

Lemma 5.4. Let $<$ be a well ordering of Ω. $M(\Omega,R,<) \cap K_2(\Omega,R) = 1$.

Proof. Let $X \in M(\Omega,R,<) \cap K_2(\Omega,R)$ and suppose that $X \neq 1$. It follows that $X = s(\Lambda_1,f_1,\mu_1)\ldots s(\Lambda_k,f_k,\mu_k)$ where $\lambda_i < \Lambda_i$, the Λ_i are non-empty and $f_i(\lambda) \neq 0$, for all $\lambda \in \Lambda_i$, $i = 1,\ldots,k$. Moreover, by relations S1 and S2 we can assume that $\mu_1 < \mu_2 < \ldots < \mu_k$.

First suppose that X is such that we can pick $\alpha,\beta \in \Omega$, $\alpha \neq \beta$ such that $f_j(\beta) \neq 0$, for some $j = 1,\ldots,k$, and $\alpha > \mu_i$ for $i = 1,\ldots,k$. Then

$$[s(\Lambda_i,f_i,\mu_i),s(\alpha,1,\beta)] = \begin{cases} s(\alpha,f_i(\beta),\mu_i) & \text{if } \beta \in \Lambda_i \\ 1 & \text{otherwise} \end{cases}$$

so that

$$Y = [X,s(\alpha,1,\beta)] = \prod_{i=1}^{k} s(\alpha,f_i(\beta),\mu_i)$$

and hence

$$Z = [Y,s(\mu_j,-1,\delta)] = s(\alpha,f_j(\beta),\delta) \qquad \delta \in \Omega,\ \delta \neq \mu_j,\alpha.$$

But $\varphi(Z) = 1$ shows that $f_j(\beta) = 0$, contrary to the choice of β.

Next suppose that $X \neq 1$ and that X is such that we cannot pick $\alpha,\beta \in \Omega$ as above. Then α is the greatest element of Ω, $\Lambda_1 = \ldots = \Lambda_k = \{\alpha\}$ and there is no $\lambda \in \Omega$ such that $\mu_k < \lambda < \alpha$. Since Ω is infinite, we can pick a subset $\{\lambda_1,\ldots,\lambda_k\}$ $\lambda_i \neq \mu_k\ \forall i,j$ of Ω such that there exists $\lambda \in \Omega$, such that $\alpha > \lambda > \lambda_i$, $i = 1,\ldots,k$. Define $w_i = w_{\lambda_i\mu_i}(1)$ and put $w = w_1\ldots w_k$. Our argument above shows that $X^w = 1$ and hence $X = 1$. This second contradiction completes the proof of the lemma.

It is well known that $KF_2(N,R)$ is the centre of $StF(N,R)$. Our next result shows that this central property holds for $KF_2(\Omega,R)$, for any choice of infinite Ω.

Proposition 5.1. $KF_2(\Omega,R) = Z(StF(\Omega,R))$.

Proof. Let $Z = Z(StF(\Omega,R))$ and $K = KF_2(\Omega,R)$. Since $EF(\Omega,R)$ has trivial centre, it is clear that $Z \leq K$. Let $X \in K$, say $X = s(\lambda_1,y_1,\mu_1)\ldots s(\lambda_k,y_k,\mu_k)$. We shall show that $X \in Z$. Let $<$ be a well ordering of Ω such that there exists $\alpha \in \Omega$ such that $\alpha > \lambda_i, \mu_i$, $i = 1,\ldots,k$. Let S_α be the subgroup generated by $\{s(\alpha,x,\beta) : \beta \in \Omega, \beta < \alpha, x \in R\}$. It is clear that S_α is abelian since

$$[s(\alpha,x,\beta),s(\alpha,y,\gamma)] = 1$$

whenever $x,y \in R$, $\beta,\gamma \in \Omega$, $\beta,\gamma < \alpha$. Moreover, each element of S_α can be written in the form $s(\alpha,x_1,\beta_1)\ldots s(\alpha,x_r,\beta_r)$ for $\beta_i < \alpha$, β_i all distinct. This factorization and the commutator identity

$$[s(\alpha,x_i,\beta_i),s(\lambda_j,y_j,\mu_j)] = \begin{cases} 1 & \lambda_j \neq \beta_i \\ s(\alpha,-y_jx_i,\mu_j) & \lambda_j = \beta_i \end{cases}$$

shows that X normalizes S_α. But $X \in K$ so Lemma 5.4 shows that X commutes with each $s(\alpha,x,\beta)$ for $x \in R$, $\beta \in \Omega$, $\beta < \alpha$. Now let T_α be the subgroup generated by $\{s(\beta,x,\alpha) : \beta \in \Omega, \beta < \alpha, x \in R\}$. A similar argument shows that X commutes with each $s(\beta,x,\alpha)$, $\beta \in \Omega$, $\beta < \alpha$, $x \in R$; for, any member of T_α can be written in the form $s(\beta_1,x_1,\alpha)\ldots s(\beta_k,x_k,\alpha)$, $\beta_i < \alpha$, β_i all distinct. As before, X normalizes T_α and if $Y \in T_\alpha$ then

$$[X,Y] = \prod_{i=1}^{r} s(\beta_i,x_i,\alpha)$$

for some $x_i \in R$ and $\beta_i \in \Omega$, $\beta_i < \alpha$. But then $\varphi([X,Y])$ is a lower triangular matrix and since $X \in K$, $x_i = 0$, for all $i = 1,\ldots,r$. Hence $[X,Y] = 1$. We have thus established so far that X commutes with $s(\beta,x,\alpha)$ and $s(\alpha,x,\beta)$ for all $x \in R$ and $\beta \in \Omega$, $\beta < \alpha$. Since

$s(\beta,x,\gamma) = [s(\beta,x,\alpha),s(\alpha,-1,\gamma)]$ we see that $X \in Z$; for, if Ω has a greatest element α_0, say, take $\alpha = \alpha_0$, otherwise we can choose α such that not only is $\alpha > \mu_i$, but also $\alpha > \beta,\gamma$.

We have now established a collection of results indicating how the $w_{\alpha\beta}(x)$, $s(\Lambda,f,\mu)$ and $h_{\lambda\mu}(x)$ interact and have given some elementary properties of $KF_2(\Omega,R)$ and $K_2(\Omega,R)$. We shall now make a few definitions which will be necessary to our proof of the main theorem of this chapter.

Definition 5.5. For every Steinberg symbol $s(\Lambda,f,\mu)$ we introduce a new symbol $s'(\Lambda,f,\mu)$ and for each of the symbols $w_{\lambda\mu}(x)$, $h_{\lambda\mu}(x)$ and $\{y,z\}_{\lambda\mu}$ introduce new symbols $w'_{\lambda\mu}(x)$, $h'_{\lambda\mu}(x)$ and $\{y,z\}'_{\lambda\mu}$ respectively, defined in terms of the $s'(\Lambda,f,\mu)$ in a way analogous to the definition of the $w_{\lambda\mu}(x)$, $h_{\lambda\mu}(x)$ and $\{y,z\}_{\lambda\mu}$.

Let E1 and E2 be the relations obtained from S1 and S2 by replacing $s(\Lambda,f,\mu)$ by $s'(\Lambda,f,\mu)$ and we define two new relations E3 and E4 by

E3: If $x,y \in R^*$ and $\lambda,\mu,\alpha,\beta \in \Omega$ are all distinct then

$$\{x,y\}'_{\lambda\mu} = \{x,y\}'_{\lambda\alpha}$$

and

$$\{x,y\}'_{\lambda\mu} = \{x,y\}'_{\alpha\beta}h'_{\lambda\alpha}([x^{-1},y^{-1}])$$

E4: $$\prod_{j=1}^{s} \{x_j,y_j\}'^{\,\epsilon_j}_{\lambda\mu} = 1 \text{ if } \prod_{j=1}^{s} [x_j^{-1},y_j^{-1}]^{\epsilon_j} = 1$$

for $\epsilon_j = \pm 1$, $x_j,y_j \in R^*$, $j = 1,\ldots,s$ and $\lambda,\mu \in \Omega$, $\lambda \neq \mu$.

Definition 5.6. We define $G(\Omega,R)$ to be the group generated by all the symbols $s'(\Lambda,f,\mu)$ for $\Lambda \subset \Omega$, $\mu \in \Omega-\Lambda$, $f:\Lambda \to R$ subject to the relations E1, E2, E3 and E4.

It is immediate that there exist epimorphisms

$$\varphi_1 : St(\Omega,R) \to G(\Omega,R) \quad \text{and} \quad \varphi_2 : G(\Omega,R) \to E(\Omega,R)$$

such that $\varphi_2 \circ \varphi_1 = \varphi$ and in particular

$$\begin{aligned}
\varphi_1(s(\Lambda,f,\mu)) &= s'(\Lambda,f,\mu), & \varphi_2(s'(\Lambda,f,\mu)) &= t(\Lambda,f,\mu),\\
\varphi_1(w_{\lambda\mu}(x)) &= w'_{\lambda\mu}(x), & \varphi_2(w'_{\lambda\mu}(x)) &= f_{\lambda\mu}(x),\\
\varphi_1(h_{\lambda\mu}(x)) &= h'_{\lambda\mu}(x), & \varphi_2(h'_{\lambda\mu}(x)) &= d_{\lambda\mu}(x),\\
\varphi_1(\{y,z\}_{\lambda\mu}) &= \{y,z\}'_{\lambda\mu}, & \varphi_2(\{y,z\}'_{\lambda\mu}) &= c(y,z)_{\lambda\mu}.
\end{aligned}$$

We shall obtain our presentation for $E(\Omega,R)$ by showing that φ_2 is an iso ~~epi~~morphism. In order to do this we make the following definition.

Definition 5.7. Define W to be the subgroup of $St(\Omega,R)$ generated by $\{w_{\lambda\mu}(x) : x \in R^*, \lambda,\mu \in \Omega, \lambda \neq \mu\}$ and for any $\lambda,\mu \in \Omega$, $\lambda \neq \mu$ define $B(\lambda,\mu)$ to be the subgroup of $St(\Omega,R)$ generated by $\{\{x,y\}_{\lambda\mu} : x,y \in R^*\}$.

Our immediate aim is to show that, when R is a division ring, $K_2(\Omega,R) \leq W$. To do this we need the following two lemmas.

Lemma 5.5. Let R be a division ring, $X \in StF(\Omega,R)$ and Ω be well ordered by $<$. There exist $M_1, M_2 \in MF(\Omega,R,<)$ and $w \in W$ such that $X = M_1 w M_2$.

Proof. Suppose that $X = s(\lambda_1,x_1,\mu_1)\ldots s(\lambda_k,x_k,\mu_k)$. Define $A = \{\alpha_i : 1 \leq i \leq \ell\}$ for $2 \leq \ell \leq 2k$ to be the set of all the λ_i, μ_i,

$i = 1,\ldots,k$, ordered by $<$ such that $\alpha_i < \alpha_{i+1}$, $i = 1,\ldots,\ell-1$. (We shall say that X is indexed by A.) From Lemma 5.2 we see that

$$s(\alpha_i,x,\alpha_j)^{w_{\alpha_i\alpha_{i+1}}(1)} = s(\alpha_{i+1},x,\alpha_j), \quad j \neq i+1$$

and by noting that there are only finitely many α_j between λ_i and μ_i it follows that X is a product of symbols $s(\alpha_i,x,\alpha_{i\pm 1})$, for $x \in R$. Moreover

$$s(\alpha_{i-1},x,\alpha_i)^{w_{\alpha_{i-1}\alpha_i}(1)} = s(\alpha_i,x,\alpha_{i-1})$$

shows that X can be expressed as a product of $s(\alpha_i,y,\alpha_{i-1})$ and of conjugates of $s(\alpha_i,y,\alpha_{i-1})$ by $w_{\alpha_{i-1}\alpha_i}(1)$, for $y \in R$. Notice that each $s(\alpha_i,y,\alpha_{i-1}) \in MF(\Omega,R,<)$ since $\alpha_{i-1} < \alpha_i$. We show that $X = M_1wM_2$, for $M_1,M_2 \in MF(\Omega,R,<)$, where M_1 and M_2 are indexed by A, and $w \in W$. To do this it will be sufficient to show that a product of the form M_1wM_2 is closed under right multiplication by $w_{\alpha\beta}(1)$, where $\alpha < \beta$ and α is adjacent to β in A in the ordering given by $<$.

Let $M_2 = s(\gamma_1,x_1,\delta_1)\ldots s(\gamma_k,x_k,\delta_k)$ for $\delta_i < \gamma_i$ and $\delta_i,\gamma_i \in A$, $i = 1,\ldots,k$. Since

$$[s(\gamma_i,x_i,\delta_i),s(\gamma_j,x_j,\delta_j)] \in MF(\Omega,R,<)$$

whenever $\gamma_i,\delta_i,\gamma_j,\delta_j \in A$, $\gamma_i > \delta_i$, $\gamma_j > \delta_j$, $x_i,x_j \in R$ we see that

$M_2 = s(\beta,y,\alpha)M_2'$ for some $y \in R$ and $M_2' \in MF(\Omega,R,<)$ such that there is no occurence of $s(\beta,*,\alpha)$ in the Steinberg symbol decomposition of M_2'. But then $M_2w_{\alpha\beta}(1) = s(\beta,y,\alpha)w_{\alpha\beta}(1)M_2''$, for some $M_2'' \in MF(\Omega,R,<)$ indexed by A, since α and β are adjacent in A. It remains to investigate $ws(\beta,y,\alpha)w_{\alpha\beta}(1)$. Let π be the permutation associated with w. If $\pi(\beta) > \pi(\alpha)$ then

$$ws(\beta,y,\alpha)w_{\alpha\beta}(1) = s(\pi(\beta),y',\pi(\alpha))ww_{\alpha\beta}(1)$$

and so $M_1wM_2w_{\alpha\beta}(1)$ has the required form. If $\pi(\beta) < \pi(\alpha)$ then notice that

$$ws(\beta,y,\alpha)w_{\alpha\beta}(1) = ww_{\alpha\beta}(y)s(\beta,-y,\alpha)s(\alpha,y^{-1},\beta)w_{\alpha\beta}(1)$$
$$= s(\pi(\alpha),y',\pi(\beta))ww_{\alpha\beta}(y)w_{\alpha\beta}(1)s(\beta,y^{-1},\alpha)$$

for some $y' \in R^*$ and so again $M_1wM_2w_{\alpha\beta}(1)$ has the reqired form: this establishes the lemma.

The proof of Lemma 5.5 followed the arguments of [30] closely. We now extend these arguments to prove

Lemma 5.6. Let R be a division ring, $X \in St(\Omega,R)$ and let Ω be well ordered by $<$. There exist $M_1, M_2 \in M(\Omega,R,<)$ and $w \in W$ such that $X = M_1wM_2$.

Proof. If $X \in St(\Omega,R)$ then $X = s(\Lambda_1',f_1',\mu_1)\ldots s(\Lambda_k',f_k',\mu_k)$. Given $s(\Lambda,f,\mu) \in St(\Omega,R)$ there exist $\Lambda_1,\Lambda_2 \subseteq \Lambda$ such that $\Lambda_1 \cup \Lambda_2 = \Lambda$, $\Lambda_1 \cap \Lambda_2 = \phi$ and $\mu < \Lambda_1$ and $\mu > \Lambda_2$. Since $s(\Lambda,f,\mu) = s(\Lambda_1,f_1,\mu)s(\Lambda_2,f_2,\mu)$ we may assume that in the decomposition of X above either $\mu_i < \Lambda_i'$ or $\mu_i > \Lambda_i'$, for $i = 1,\ldots,k$. Let λ_i denote the least element of Λ_i', $i = 1,\ldots,k$ and let $A = \{\alpha_i : 1 \leq i \leq \ell\}$, $2 \leq \ell \leq 2k$ be the set of all the λ_i,μ_i ordered by $<$ such that $\alpha_i < \alpha_{i+1}$. Suppose that $\mu_i > \Lambda_i'$. Then in particular $\mu_i > \lambda_i$ and if $\mu_i = \alpha_{j+1}$ and $\lambda_i = \alpha_j$ then

$$s(\Lambda_i',f_i',\mu_i)^{w_{\alpha_j\alpha_{j+1}}(1)} = s(\Lambda_i,g,\lambda_i)$$

for some $\Lambda_i \subset \Omega$, $g:\Lambda_i \to R$ and $\lambda_i < \Lambda_i$. If $\mu_i = \alpha_k$ and $\lambda_i = \alpha_j$ with $k > j+1$ then notice that

$$s(\Lambda_i', f_i, \mu_i)^{w_{\alpha_{k-1}\alpha_k}(1)} = s(\Lambda_i, g, \alpha_{k-1})$$

for some $\Lambda_i \subset \Omega$, $g:\Lambda_i \to R$ and where λ_i is the least element of Λ_i. If $\alpha_{k-1} \nless \Lambda_i$ then we can partition Λ_i as $\Lambda_{i1} \cup \Lambda_{i2}$ so that $s(\Lambda_i', f_i, \mu_i)^{w_{\alpha_{k-1}\alpha_k}(1)}$ is a product of $s(\Lambda_{i1}, g_1, \alpha_{k-1})$ and $s(\Lambda_{i2}, g_2, \alpha_{k-1})$ where $\alpha_{k-1} > \Lambda_{i1}$, the least element of Λ_{i1} is λ_i and $\alpha_{k-1} < \Lambda_{i2}$.

Continuing in this way, we see that X can be expressed as a product of the symbols $s(\Lambda_i, f_i, \alpha_i)$ with $\alpha_i < \Lambda_i$, $\alpha_i \in A$, $\Lambda_i \subset \Omega$, $f_i:\Lambda_i \to R$ and the symbols $w_{\alpha_i\alpha_{i+1}}(1)$ and $w_{\alpha_i\alpha_{i+1}}(-1)$. Thus, since each $s(\Lambda_i, f_i, \alpha_i) \in M(\Omega, R, <)$, to show that $X = M_1wM_2$ it will be sufficient to show that the form M_1wM_2, for any $M_1, M_2 \in M(\Omega, R, <)$ and $w \in W$, is closed with respect to right multiplication by $w_{\alpha_i\alpha_{i+1}}(-1)$. Let π be the permutation associated with w. We can write M_2 as

$$M_2 = \prod_{j=1}^{k} s(\alpha_{i+1}, x_j, \lambda_j) s(\Lambda_i, f_i, \alpha_i) s(\Lambda_{i+1}, f_{i+1}, \alpha_{i+1}) M_4$$

for some $k \in N$, $x_j \in R$ and $\lambda_j < \alpha_{i+1}$ and where $\alpha_{i+1} \notin \Lambda_i$ and with

$$M_4 = \prod_{j=1}^{r} s(\Phi_j, g_j, \varphi_j)$$

for some $r \in N$, $g_j:\Phi_j \to R$, $\varphi_j < \Phi_j$, $\alpha_{i+1} \notin \Phi_j$ and $\varphi_j \neq \alpha_i, \alpha_{i+1}$, $j = 1, \ldots, r$. We see that

$$s(\Lambda_{i+1}, f_{i+1}, \alpha_{i+1}) M_4 w_{\alpha_i\alpha_{i+1}}(1) = w_{\alpha_i\alpha_{i+1}} M_4'$$

for some $M_4' \in M(\Omega, R, <)$. It remains to investigate

$$w \prod_{j=1}^{k} s_j s(\Lambda_i, f_i, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1)$$

where $s_j = s(\alpha_{i+1}, x_j, \lambda_j)$. (We can assume that $\alpha_{i+1} > \Lambda_i$, for otherwise we can write $\Lambda_i = \Lambda_i' \cup \Lambda_i''$, where $\alpha_{i+1} < \Lambda_i'$ and $\alpha_{i+1} > \Lambda_i''$ and notice that

$$s(\Lambda_i', f_i', \alpha_i) w_{\alpha_i\alpha_{i+1}}(1) = w_{\alpha_i\alpha_{i+1}}(1)t$$

for some $t \in M(\Omega, R, <)$.) Partition Λ_i as $\Psi_1 \cup \Psi_2 \cup (\Lambda_f \cap \Lambda_i)$ where Λ_f is the set of elements of Ω moved by π together with α_{i+1} and $\pi(\alpha_i) < \Psi_1$, $\pi(\alpha_i) > \Psi_2$. Then Ψ_1 and Ψ_2 are fixed by π and

$$s(\Lambda_i, f_i, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1) = s(\Psi_1, g_1, \alpha_i) s(\Lambda_f \cap \Lambda_i, g_2, \alpha_i) w^{-1} w s(\Psi_2, g, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1)$$

for $g_1 : \Psi_1 \to R$, $g_2 : \Lambda_f \cap \Lambda_i \to R$ and $g : \Psi_2 \to R$. Notice that

$$s(\Psi_2, g, \alpha_i) = w_{\alpha_i\lambda}(x) s(\Psi_0, x^{-1}g, \lambda) s(\lambda, -x, \alpha_i) s(\alpha_i, x^{-1}, \lambda)$$

where $\Psi_0 = \Psi_2 - \{\lambda\}$ and λ is the least element of Ψ_2 and $x = g(\lambda)$. Then

$$\begin{aligned} & ws(\Psi_2, g, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1) \\ = \; & ww_{\alpha_i\lambda}(x) s(\Psi_0, x^{-1}g, \lambda) s(\lambda, -x, \alpha_i) s(\alpha_i, x^{-1}, \lambda) w_{\alpha_i\alpha_{i+1}}(1) \\ = \; & ww_{\alpha_i\lambda}(x) s(\lambda, -x, \alpha_i) s(\Psi_0, g, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1) s(\Psi_0, x^{-1}g, \lambda) s(\alpha_{i+1}, x^{-1}, \lambda) \\ = \; & s(\pi(\alpha_i), x', \lambda) s(\Psi_0, \cancel{g}\, {-x^{-1}g}, \lambda) ww_{\alpha_i\lambda}(x) w_{\alpha_i\alpha_{i+1}}(1) M_0 \end{aligned}$$

for some $M_0 \in M(\Omega, R, <)$. Hence

$$\begin{aligned} & w \prod_{j=1}^{k} s_j s(\Lambda_i, f_i, \alpha_i) w_{\alpha_i\alpha_{i+1}}(1) \\ = \; & w \prod_{j=1}^{k} s_j s(\Psi_1, g_1, \alpha_i) s(\Lambda_f \cap \Lambda_i, g_2, \alpha_i) w^{-1} s(\pi(\alpha_i), x', \lambda) s(\Psi_0, \cancel{g}\, {-x^{-1}g}, \lambda) \cdot \\ & ww_{\alpha_i\lambda}(x) w_{\alpha_i\alpha_{i+1}}(1) M_0 . \end{aligned}$$

But $s(\Psi_0, g, \lambda)$ commutes with $s(\Lambda_f \cap \Lambda_i, g_2, \alpha_i) w^{-1} s(\pi(\alpha_i), x', \lambda)$ since $\Psi_0 \cup \{\lambda\}$ is disjoint from Λ_f and since $s(\Psi_1, g_1, \alpha_i)$ and $s(\Psi_0, g, \lambda)$ commute with the s_j modulo $StF(\Omega,R)$ we see that

$$w \prod_{j=1}^{k} s_j s(\Lambda_i, f_i, \alpha_i) w_{\alpha_i \alpha_{i+1}}(1) = s(\Psi_1, g'_1, \pi^{-1}(\alpha_i)) s(\Psi_0, \overset{\pi^{-1}g}{\cancel{g}}, \lambda) X M_0$$

for some $X \in StF(\Omega,R)$ and $M_0 \in M(\Omega,R,<)$. Thus

$$M_1 w M_2 w_{\alpha_i \alpha_{i+1}}(1) = M_1 s(\Psi_1, g'_1, \pi^{-1}(\alpha_i)) s(\Psi_0, \overset{\pi^{-1}g}{\cancel{g}}, \lambda) X M_0 .$$

However $X = M'_1 w' M'_2$, for some $w \in W$ and $M'_1, M'_2 \in M(\Omega,R,<)$ and thus we see that $M_1 w M_2 w_{\alpha_i \alpha_{i+1}}(1)$ has the required form.

Corollary 5.2. When R is a division ring, $K_2(\Omega,R) \leq W$.

Proof. Let $X \in K_2(\Omega,R)$ and well order Ω by $<$. From the lemma we know that $X = M_1 w M_2$, for $M_1, M_2 \in M(\Omega,R,<)$ and $w \in W$. But $\varphi(X) = 1$ shows that $\varphi(w) = \varphi(M_1^{-1} M_2^{-1})$. However $\varphi(M_1^{-1} M_2^{-1})$ is an upper triangular matrix and unless $\varphi(w) = 1$, $\varphi(w)$ has non-zero entries below the diagonal; thus $w \in K_2(\Omega,R)$ and $M_1 = M_2^{-1}$ since $M(\Omega,R,<) \cap K_2(\Omega,R) = 1$. We now assert that w and M_1 commute. Let π be the permutation associated with w and let Λ_f be the set of elements of Ω moved by π. We can write

$$M_1 = s(\Lambda_1, f_1, \mu_1) \ldots s(\Lambda_r, f_r, \mu_r)$$

where $\mu_i < \Lambda_i$, $i = 1, \ldots, r$, the μ_i are all distinct and $\mu_i < \mu_j$ whenever $i < j$. Consider the product

$$s(\Lambda_i, f_i, \mu_i) s(\Lambda_{i+1}, f_{i+1}, \mu_{i+1})$$

and suppose that $\Lambda_{i+1} \cap \Lambda_f = \Lambda'_{i+1}$ and put $\Lambda_{i+1} - \Lambda'_{i+1} = \Lambda''_{i+1}$. Then

$$\begin{aligned}
& s(\Lambda_i, f_i, \mu_i) s(\Lambda_{i+1}, f_{i+1}, \mu_{i+1}) \\
= \; & s(\Lambda'_{i+1}, f'_{i+1}, \mu_{i+1}) s(\Lambda_i, f_i, \mu_i) [s(\Lambda_i, f_i, \mu_i), s(\Lambda'_{i+1}, f'_{i+1}, \mu_{i+1})] s(\Lambda''_{i+1}, f''_{i+1}, \mu_{i+1}) \\
= \; & s(\Lambda'_{i+1}, f'_{i+1}, \mu_{i+1}) s(\Lambda_i, f_i, \mu_i) s(\Lambda'_{i+1}, f_i(\mu_{i+1}) f'_{i+1}, \mu_i) s(\Lambda''_{i+1}, f''_{i+1}, \mu_{i+1}).
\end{aligned}$$

By repeating this argument we see that we can write $M_1 = M_f M_0$ where $M_f \in MF(\Omega, R, \triangleleft)$ and

$$M_0 = \prod_{i=1}^{k} s(\Phi_i, g_i, \alpha_i)$$

where $\Lambda_f \cap \{\bigcup_{i=1}^{k} \Phi_i\}$ is empty and $\alpha_i < \Phi_i$, $i = 1,\ldots,k$. Now consider the product $wM_f M_0$. Since $w \in KF_2(\Omega,R) = Z(StF(\Omega,R))$ it follows that $wM_f = M_f w$. If $\alpha_i \notin \Lambda_f$, since ~~since~~ $\Lambda_f \cap \Phi_i$ is empty we see that $ws(\Phi_i, g_i, \alpha_i) = s(\Phi_i, g_i, \alpha_i)w$. If $\alpha_i \in \Lambda_f$ then pick $\alpha \in \Omega$, $\alpha \neq \alpha_i$, $\alpha \notin \Phi_i$, $\alpha \notin \Lambda_f$ and note that

$$s(\Phi_i, g_i, \alpha_i) = s(\Phi_i, g_i, \alpha)^{w_{\alpha\alpha_i}(1)}$$

so that

$$\begin{aligned} ws(\Phi_i, g_i, \alpha_i) &= ws(\Phi_i, g_i, \alpha)^{w_{\alpha\alpha_i}(1)} \\ &= s(\Phi_i, g_i, \alpha)^{w_{\alpha\alpha_i}(1)} w \\ &= s(\Phi_i, g_i, \alpha_i) w \end{aligned}$$

since $w \in KF_2(\Omega,R) = Z(StF(\Omega,R))$ and $(\Phi_i \cup \{\alpha\}) \cap \Lambda_f$ is empty. (If $\Omega = \Lambda_f \cup \{\alpha_i\} \cup \Phi_i$ then we can write Φ_i as the disjoint union of two subsets and proceed in two stages.) By repeating this argument we see that $wM_0 = M_0 w$ and conclude that $M_1 w M_1^{-1} = w$. This completes the proof of the corollary.

We have shown that when R is a division ring $K_2(\Omega,R)$ is contained in W. We now show that $K_2(\Omega,R)$ can be 'narrowed down' even further.

Definition 5.8. We define H to be the subgroup of $St(\Omega,R)$ generated by $\{h_{\lambda\mu}(x) : x \in R^*,\ \lambda,\mu \in \Omega,\ \lambda \neq \mu\}$.

Lemma 5.7. When R is a division ring, $K_2(\Omega,R) \leq H$.

Proof. Let $X \in K_2(\Omega,R)$. It follows from the corollary that $X = w_{\lambda_1\mu_1}(x_1)\ldots w_{\lambda_k\mu_k}(x_k)$. Well order Ω by $<$ and let α be the least of all the λ_i, μ_i. From the proof of Lemma 5.3 we see that

$$w_{\varphi\lambda}(1) = h_{\varphi\lambda}(x)^{-1} w_{\varphi\lambda}(x) \qquad \text{for any } \varphi,\lambda \in \Omega,\ \varphi \neq \lambda.$$

Thus $w_{\varphi\lambda}(1) \equiv w_{\varphi\lambda}(x) \bmod H$. Denote this common residue class by $w_{\varphi\lambda}$. It now follows from Corollary 5.1 that, for any $\lambda,\mu,\beta \in \Omega$, $\lambda \neq \mu$, $\alpha \neq \beta$

$$w_{\lambda\mu} w_{\alpha\beta} \equiv w_{\pi(\alpha\beta)} w_{\lambda\mu} \quad \bmod H$$

where π is the permutation that interchanges λ and μ and that for any $\alpha,\beta,\lambda \in \Omega$, α,β,γ all distinct

$$w_{\alpha\beta} w_{\alpha\gamma} \equiv w_{\alpha\gamma} w_{\gamma\beta} \quad \bmod H$$

and

$$w_{\alpha\gamma} w_{\alpha\gamma} \equiv 1 \quad \bmod H.$$

By using these identities we see that we can push all the $w_{\alpha *}$ that occur in the decomposition of X to the left and thus obtain only one occurence of $w_{\alpha *}$. But this cannot happen since $\varphi(X) = 1$. Repeating this argument as often as necessary we see that $X \equiv 1 \bmod H$ and this completes the proof of the lemma.

Having now shown that $K_2(\Omega,R)$ is contained in H we use this result to obtain a factorization of elements of $\varphi_1(X)$, for $X \in K_2(\Omega,R)$

in terms of elements of $\varphi_1(H)$. (The proof of Lemma 5.8 follows the arguments of Green [20].)

Lemma 5.8. Let R be a division ring, let Ω be well ordered by $<$ and let $X \in H$, say

$$X = h_{\lambda_1\mu_1}(x_1)^{\varepsilon_1} \cdots h_{\lambda_k\mu_k}(x_k)^{\varepsilon_k}$$

for $\lambda_i, \mu_i \in \Omega$, $x_i \in R$, $\varepsilon_i = \pm 1$. Let $A = \{ \alpha_i : 1 \le i \le \ell \}$, for $2 \le \ell \le 2k$, be the set of all the λ_i, μ_i ordered by $<$ such that $\alpha_i < \alpha_{i+1}$.

$$\varphi_1(X) = \varphi_1(b) \prod_{i=1}^{\ell-1} h'_{\alpha_i\alpha_{i+1}}(y_i)$$

for some $y_i \in R$ and $b \in B(\alpha_1, \alpha_2)$.

Proof. Let H_i be the subgroup generated by the $h_{\alpha_i\alpha_{i+1}}(u)$ $\alpha_i, \alpha_{i+1} \in A$, $u \in R^*$. Since

$$h_{\alpha_i\alpha_{i+1}}(u)^{h_{\alpha\beta}(x)} = w_{\alpha_i\alpha_{i+1}}(u)^{h_{\alpha\beta}(x)} w_{\alpha_i\alpha_{i+1}}(-1)^{h_{\alpha\beta}(x)}$$

and

$$w_{\alpha_i\alpha_{i+1}}(u)^{h_{\alpha\beta}(x)} = (w_{\alpha_i\alpha_{i+1}}(u)^{w_{\alpha\beta}(x)})^{w_{\alpha\beta}(-1)}$$

$$= w_{\alpha_i\alpha_{i+1}}(u') \qquad u' \in R^*$$

and

$$w_{\alpha_i\alpha_{i+1}}(-1)^{h_{\alpha\beta}(x)} = (w_{\alpha_i\alpha_{i+1}}(-1)^{w_{\alpha\beta}(x)})^{w_{\alpha\beta}(-1)}$$

$$= w_{\alpha_i\alpha_{i+1}}(v') \qquad v' \in R^*$$

and

$$w_{\alpha_i\alpha_{i+1}}(u')w_{\alpha_i\alpha_{i+1}}(v') = h_{\alpha_i\alpha_{i+1}}(u')h_{\alpha_i\alpha_{i+1}}(-v')^{-1}$$

we see that H_i is a normal subgroup of H. It follows that $H_1 \dots H_{\ell-1}$ is also a normal subgroup. We shall show that $H_1 \dots H_{\ell-1}$ contains all the $h_{\lambda_i\mu_i}(x_i)$. Suppose that $\lambda_i < \mu_i$ with $\lambda_i = \alpha_{k_1} < \dots < \alpha_{k_r} = \mu_i$. By noting that

$$h_{\lambda_i\mu_i}(x_i) = h_{\alpha_{k_1}\alpha_{k_r}}(x_i) = h_{\alpha_{k_2}\alpha_{k_r}}(x_i)h_{\alpha_{k_1}\alpha_{k_2}}(x_i)$$

from Lemma 5.3 and by extending this argument we see that $H_1 \dots H_{\ell-1}$ contains all the $h_{\lambda_i\mu_i}(x_i)$, and, indeed, $H_1 \dots H_{i-1}$ contains all the $h_{\alpha_1\alpha_i}(u)$, $i = 2,\dots,\ell$, $u \in R^*$. Thus we can write $X = h_1h_2\dots h_{\ell-1}$, for $h_i \in H_i$, $i = 1,\dots,\ell-1$. We now assert that if $i \neq 1$ $h_i \equiv hh_{\alpha_i\alpha_{i+1}}(u)$ mod K, where $K = \ker \varphi_1$, $u \in R^*$ and $h \in H_1H_2\dots H_{i-1}$; for suppose

$$h_i = h_{\alpha_i\alpha_{i+1}}(u_1)^{\varepsilon_1}\dots h_{\alpha_i\alpha_{i+1}}(u_r)^{\varepsilon_r}.$$

Since

$$\begin{aligned} h_{\alpha_i\alpha_{i+1}}(u)^{-1} &= (h_{\alpha_i\alpha_{i+1}}(u^{-1})h_{\alpha_i\alpha_{i+1}}(u)h_{\alpha_i\alpha_{i+1}}(1)^{-1})^{-1}h_{\alpha_i\alpha_{i+1}}(u^{-1}) \\ &= \{u^{-1},u\}^{-1}_{\alpha_i\alpha_{i+1}}h_{\alpha_i\alpha_{i+1}}(u^{-1}) \\ &\equiv \{u^{-1},u\}^{-1}_{\alpha_1\alpha_2}h_{\alpha_i\alpha_{i+1}}(u^{-1}) \quad \text{mod } K \end{aligned}$$

by relation E3, we can suppose that $r > 1$ and $\varepsilon_1 = 1$. If $\varepsilon_2 = 1$ we have

$$\begin{aligned} h_i &= h_{\alpha_i\alpha_{i+1}}(u_1)h_{\alpha_i\alpha_{i+1}}(u_2)h_{\alpha_i\alpha_{i+1}}(u_3)^{\varepsilon_3}\dots \\ &= h_{\alpha_i\alpha_{i+1}}(u_1)h_{\alpha_i\alpha_{i+1}}(u_2)h_{\alpha_i\alpha_{i+1}}(u_2u_1)^{-1}h_{\alpha_i\alpha_{i+1}}(u_2u_1)h_{\alpha_i\alpha_{i+1}}(u_3)^{\varepsilon_3}\dots \end{aligned}$$

and hence

$$h_i \equiv \{u_1,u_2\}_{\alpha_1\alpha_2} h_{\alpha_1\alpha_i}([u_1,u_2])^{-1} h_{\alpha_i\alpha_{i+1}}(u_3)^{\varepsilon_3}\ldots \quad \text{mod } K.$$

But $\{u_1,u_2\}_{\alpha_1\alpha_2} h_{\alpha_1\alpha_i}([u_1,u_2])^{-1} \in H_1\ldots H_{i-1}$, by our earlier remarks and so an inductive argument on r shows that h_i has the required form, when $\varepsilon_2 = 1$. Now suppose that $\varepsilon_2 = -1$. Since

$$h_{\alpha_i\alpha_{i+1}}(u_2^{-1}u_1)h_{\alpha_i\alpha_{i+1}}(u_2)h_{\alpha_i\alpha_{i+1}}(u_1)^{-1} = \{u_2^{-1}u_1,u_2\}_{\alpha_i\alpha_{i+1}}$$

we see that

$$h_{\alpha_i\alpha_{i+1}}(u_1)h_{\alpha_i\alpha_{i+1}}(u_2)^{-1} = \{u_2^{-1}u_1,u_2\}^{-1}_{\alpha_i\alpha_{i+1}} h_{\alpha_i\alpha_{i+1}}(u_2^{-1}u_1)$$

But

$$\{u_2^{-1}u_1,u_2\}_{\alpha_1\alpha_2} \equiv \{u_2^{-1}u_1,u_2\}_{\alpha_i\alpha_{i+1}} h_{\alpha_1\alpha_i}([u_2^{-1}u_1,u_2]) \quad \text{mod } K$$

so that

$$h_{\alpha_i\alpha_{i+1}}(u_1)h_{\alpha_i\alpha_{i+1}}(u_2)^{-1} \equiv (\{u_2^{-1}u_1,u_2\}_{\alpha_1\alpha_2} h_{\alpha_1\alpha_i}([u_2^{-1}u_1,u_2]))^{-1}.$$
$$h_{\alpha_i\alpha_{i+1}}(u_2^{-1}u_1) \quad \text{mod } K$$

and

$$h_i \equiv h' h_{\alpha_i\alpha_{i+1}}(u_2^{-1}u_1)h_{\alpha_i\alpha_{i+1}}(u_3)^{\varepsilon_3}\ldots \quad \text{mod } K.$$

By our remarks above, $h' \in H_1\ldots H_{i-1}$ and an inductive argument on r shows that h_i has the required form. However, we have shown that $X = h_1h_2\ldots h_{\ell-1}$, for $h_i \in H_i$. Thus, since $h_{\ell-1} \equiv hh_{\alpha_{\ell-1}\alpha_\ell}(u_\ell)$, for $h \in H_1\ldots H_{\ell-2}$ we see that

$$\begin{aligned} X &\equiv h' h_{\alpha_{\ell-1}\alpha_\ell}(u_\ell) \qquad \text{for } h' \in H_1 \ldots H_{\ell-2} \\ &\equiv h'' h_{\alpha_{\ell-2}\alpha_{\ell-1}}(u_{\ell-1}) h_{\alpha_{\ell-1}\alpha_\ell}(u_\ell) \qquad h'' \in H_1 \ldots H_{\ell-3} \\ &\quad \vdots \\ &\equiv h_0 h_{\alpha_2\alpha_3}(u_3) \ldots h_{\alpha_{\ell-1}\alpha_\ell}(u_\ell), \qquad h_0 \in H_1 . \end{aligned}$$

Thus $h_0 = h_{\alpha_1\alpha_2}(x_1)^{\varepsilon_1} \ldots h_{\alpha_1\alpha_2}(x_r)^{\varepsilon_r}$ and our argument above shows that

$h_0 \equiv b h_{\alpha_1\alpha_2}(u_2)$.

If we now take images under φ_1 we see that $\varphi_1(X)$ has the required form.

Now let us take stock of the current situation. We have epimorphisms φ_1, φ_2 with $\varphi = \varphi_2 \circ \varphi_1$ and

$$St(\Omega,R) \xrightarrow{\varphi_1} G(\Omega,R) \xrightarrow{\varphi_2} E(\Omega,R)$$

where $G(\Omega,R)$ is the group generated by the symbols $s'(\Lambda,f,\mu)$ subject to the relations E1, E2, E3 and E4. We have shown that $K_2(\Omega,R) = \ker \varphi \leq H$ and if $X \in H$ then

$$\varphi_1(X) = \varphi_1(b) \prod_{i=1}^{\ell-1} h'_{\alpha_i\alpha_{i+1}}(y_i)$$

for some $y_i \in R^*$, $\alpha_i \in \Omega$, $\alpha_i < \alpha_{i+1}$ and $b \in B(\alpha_1,\alpha_2)$. However, if $Y \in \ker \varphi_2$ then $Y = \varphi_1(X)$, for some $X \in H$, so that if

$$Y = \varphi_1(b) \prod_{i=1}^{\ell-1} h'_{\alpha_i\alpha_{i+1}}(y_i)$$

then

$$1 = c \prod_{i=1}^{\ell-1} d_{\alpha_i\alpha_{i+1}}(y_i)$$

for some $c \in \varphi(B(\alpha_1,\alpha_2))$. But $d_{\alpha_i\alpha_{i+1}}(y_i)$ is the matrix that has y_i in the α_ith diagonal position and y_i^{-1} in the α_{i+1}st diagonal position and coincides with the identity matrix elsewhere, that is

$$d_{\alpha_i\alpha_{i+1}}(y_i) = \begin{bmatrix} \ddots_1 & & & & & \\ & y_i & & & \bigcirc & \\ & & 1 & & & \\ & & & \ddots_1 & & \\ & \bigcirc & & & y_i^{-1} & \\ & & & & & 1_{\ddots} \end{bmatrix}$$

so that $\prod_{i=1}^{\ell-1} d_{\alpha_i\alpha_{i+1}}(y_i)$ is the diagonal matrix with y_1 in the α_1st entry, $y_{\ell-1}^{-1}$ in the α_ℓth entry and $y_{i-1}^{-1}y_i$ in the α_ith entry, for $1 < i < \ell$. Since c is the diagonal matrix whose only entry that does not coincide with the corresponding entries of the identity matrix is the α_1st we see that $y_1 = \dots = y_{\ell-1} = 1$. This in turn shows that $c = 1$ and $\varphi_2(Y) = c$ and $Y \in \varphi_1(B(\alpha_1,\alpha_2))$. If, then

$$Y = \prod_{j=1}^{r} \{x_j,y_j\}_{\alpha_1\alpha_2}^{\epsilon_j} \qquad x_j,y_j \in R^*,\ \epsilon_j = \pm 1$$

we see that

$$\prod_{j=1}^{r} [x_j^{-1},y_j^{-1}]^{\epsilon_j} = 1.$$

Relation E4 now shows that $Y = 1$ and we deduce that φ_2 is an isomorphism. We are now able to state

Theorem 5.1. If R is a division ring and Ω is an infinite set then $E(\Omega,R)$ is isomorphic to the group generated by all the symbols $s'(\Lambda,f,\mu)$, $\Lambda \subset \Omega$, $\mu \in \Omega-\Lambda$, $f:\Lambda \to R$, subject to the relations E1, E2, E3 and E4.

To conclude chapter five we shall show how we may also obtain a presentation for ~~St~~ E $F(\Omega,R)$, for any infinite set Ω and division ring R. Define the relations SF1 and SF2 by

SF1: for $x,y \in R$, $\lambda,\mu \in \Omega$, $\lambda \neq \mu$, $s(\lambda,x,\mu)s(\lambda,y,\mu) = s(\lambda,x+y,\mu)$

SF2: for $x,y \in R$, $\lambda,\mu,\alpha,\beta \in \Omega$, $\lambda \neq \mu$, $\alpha \neq \beta$

$$[s(\lambda,x,\mu),s(\alpha,y,\beta)] = \begin{cases} s(\lambda,-yx,\beta) & \text{if } \mu = \alpha \\ s(\alpha,xy,\mu) & \text{if } \beta = \lambda \\ 1 & \text{otherwise} \end{cases}$$

For each finite Steinberg symbol $s(\lambda,x,\mu)$ introduce a symbol $s'(\lambda,x,\mu)$. Let $GF(\Omega,R)$ be the group generated by all the symbols $s'(\lambda,x,\mu)$ for $\lambda,\mu \in \Omega$, $\lambda \neq \mu$, $x \in R$ subject to the relations EF1, EF2, E3 and E4, where EF1 and EF2 are obtained from SF1 and SF2 by replacing the $s(\lambda,x,\mu)$ by the $s'(\lambda,x,\mu)$. It follows that, given the epimorphisms

$$StF(\Omega,R) \xrightarrow{\varphi_1'} GF(\Omega,R) \xrightarrow{\varphi_2'} EF(\Omega,R)$$

$$s(\lambda,x,\mu) \to s'(\lambda,x,\mu) \to t(\lambda,x,\mu)$$

$\ker \varphi_2' = 1$, when R is a division ring. We obtain

Theorem 5.2. If R is a division ring and Ω is an infinite set then $EF(\Omega,R)$ is isomorphic to the group generated by all the symbols $s'(\lambda,x,\mu)$, $\lambda,\mu \in \Omega$, $\lambda \neq \mu$, $x \in R$, subject to the relations EF1, EF2, E3 and E4.

In particular, we see that Theorem 5.2 gives a presentation for $EF(N,R)$, the derived group of the stable general linear group of Bass [4].

Chapter six

Throughout this chapter, R shall always denote a ring with identity, Ω an infinite set and M the free R-module $R^{(\Omega)}$. In chapter three we were concerned with the problem of locating and classifying the normal subgroups of $GL(\Omega,R)$. We found that each normal subgroup H determined a unique ideal p of R - called the level of H - such that

$$E[\Omega,p] \leq H \leq GL'(\Omega,p) \tag{*}$$

If R was d-finite then $E[\Omega,p]$ coincided with the elementary subgroup $E(\Omega,p)$. If $X \in E(\Omega,p)$ then, regarded as an $\Omega \times \Omega$ matrix, X differs from I in only finitely many rows. In some sense, this means that the group $E(\Omega,p)$ is a 'small' subgroup of $GL(\Omega,R)$ and we shall now attempt to improve the sandwich (*) by enlarging the lower bound as much as possible. At the same time we shall try to make the sandwich as thin as possible by making the upper bound smaller. In the sequel, underlined lower case letters shall denote cardinals.

Let $\underline{w}$ = card Ω.

Definition 6.1. For any infinite cardinal $\underline{a} \leq \underline{w}$ denote by $GL(\Omega,R,\underline{a})$ the subgroup of $GL(\Omega,R)$ generated by those $X \in GL(\Omega,R)$ such that X-I differs from the zero matrix in less than $\underline{a}$ rows.

Definition 6.2. For any $\underline{a} \leq \underline{w}$ and any two sided ideal p of R define $GL(\Omega,p,\underline{a})$ to be the subgroup of $GL(\Omega,R)$ generated by those $X \in GL(\Omega,R)$ which are such that (X-I) differs from the zero matrix in less that $\underline{a}$ rows and such that $J(X) \leq p$. (Since $X_{\alpha\alpha} = 1$ for some $\alpha \in \Omega$ the condition that $J(X) \leq p$ is equivalent to the condition that

$(X\text{-}I)_{\alpha\beta} \in p$, for all $\alpha, \beta \in \Omega$.)

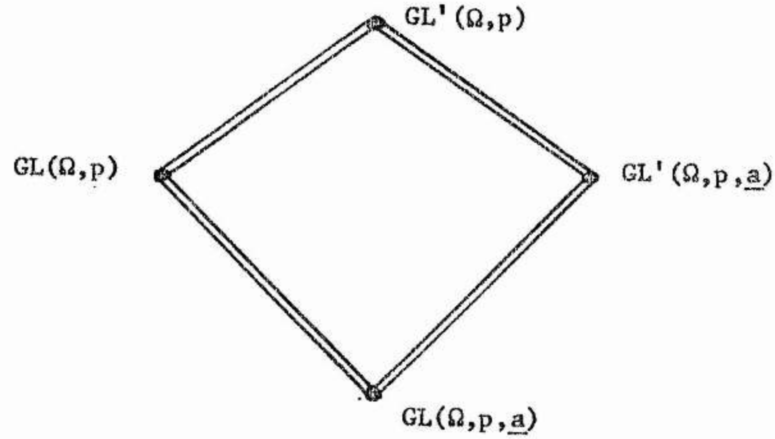
For any $X \in GL(\Omega, R, \underline{a})$, $J(X)$ is the least ideal p of R such that $X \in GL(\Omega, p, \underline{a})$. Moreover, we see that for every infinite cardinal $\underline{a} \leq \underline{w}$ and ideal p of R, the group $GL(\Omega, p, \underline{a})$ is a normal subgroup of $GL(\Omega, R)$. This is because:

(i) if $X \in GL(\Omega, p, \underline{a})$ and $Y \in GL(\Omega, R)$, $X^Y - I$ differs from the zero matrix in less than $\underline{a}$ rows since Y is column finite;

(ii) by Lemma 3.1, if $X \in GL(\Omega, p, \underline{a})$ and $Y \in GL(\Omega, R)$ then $J(X^Y) = J(X)$.

Definition 6.3. Let $D(R)$ denote the set of $\Omega \times \Omega$ matrices of the form rI, where r is a central unit of R. (We see that $D(R) = Z(GL(\Omega, R))$ from chapter three.) For every infinite $\underline{a} \leq \underline{w}$ and every two sided ideal p of R we define $GL'(\Omega, p, \underline{a})$ to be the group $D(R)GL(\Omega, p, \underline{a})$.

It is clear from our remarks above that $GL'(\Omega, p, \underline{a})$ is a normal subgroup of $GL(\Omega, R)$ and that the groups defined satisfy the relations implied by the following diagram.



Definition 6.4. Let $\Lambda, \Phi \subset \Omega$, card $\Phi = \underline{a} < \underline{w}$. We say that (Λ, Φ, i) is an elementary correspondence in Ω if and only if Λ and Φ are disjoint and for each $\lambda \in \Lambda$, $i(\lambda)$ is a finite subset of Φ such that

$$\bigcup_{\lambda \in \Lambda} i(\lambda) = \Phi.$$

Let $\mathcal{E}$ denote the set of all elementary correspondences in Ω. Given $(\Lambda, \Phi, i) \in \mathcal{E}$, a mapping $f: \Phi \times \Lambda \to R$ is said to be an elementary mapping if $f(\varphi, \lambda) \neq 0$ only if $\varphi \in i(\lambda)$. We shall adopt the convention that for $(\Lambda, \Phi, i) \in \mathcal{E}$, an elementary mapping $f: \Phi \times \Lambda \to R$ extends to a mapping $f: \Omega \times \Omega \to R$ by defining $f(\varphi, \lambda) = 0$, whenever $\lambda \notin \Lambda$ or $\varphi \notin \Phi$. For every elementary correspondence (Λ, Φ, i) and any elementary mapping $f: \Phi \times \Lambda \to R$ define the automorphism $t(\Lambda, \Phi, i, f)$ of M by:

$$t(\Lambda, \Phi, i, f)e_\omega = e_\omega + \sum_\varphi e_\varphi f(\varphi, \omega), \quad \omega \in \Omega .$$

$t(\Lambda, \Phi, i, f)$ can be regarded as an $\Omega \times \Omega$ column finite matrix with non-trivial entries only in the rows indexed by Φ and in the columns indexed by Λ and with the property that if the λth row is non-trivial then the λth column is trivial and *vice versa*. For this reason, we shall call $t(\Lambda, \Phi, i, f)$ an elementary matrix. If we write $t(\Lambda, \Phi, i, f) = I + E$ then we see that $E^2 = 0$.

Definition 6.5. For any infinite $\underline{a} \leq \underline{w}$ we define the group $E(\Omega, R, \underline{a})$ to be the subgroup of $GL(\Omega, R)$ generated by the elementary matrices $t(\Lambda, \Phi, i, f)$ for all $(\Lambda, \Phi, i) \in \mathcal{E}$, with card $\Phi < \underline{a}$ and all elementary mappings $f: \Phi \times \Lambda \to R$.

It is clear that when $\underline{a} = \aleph_0$, $E(\Omega, R, \underline{a})$ is just the group $E(\Omega, R)$ of chapter three.

Definition 6.6. For any infinite $\underline{a} \leq \underline{w}$ and any two sided ideal p of R we define $E(\Omega,p,\underline{a})$ to be the normal subgroup of $E(\Omega,R,\underline{a})$ generated by all the elementary matrices $t(\Lambda,\Phi,i,f)$, for $(\Lambda,\Phi,i) \in \mathcal{E}$ with card $\Phi < \underline{a}$ and all elementary mappings $f:\Phi \times \Lambda \to p$.

We see that, for any infinte $\underline{a} \leq \underline{w}$ and two sided ideal p of R, $E(\Omega,p,\underline{a}) \leq GL'(\Omega,p,\underline{a})$ and whenever $X \in E(\Omega,p,\underline{a})$, $J(X) \leq p$. We shall see that when R is d-finite, it is the groups $E(\Omega,p,\underline{a})$ and $GL'(\Omega,p,\underline{b})$ for infinite $\underline{a},\underline{b} \leq \underline{w}$ that provide the sandwich for the normal subgroups of $GL(\Omega,R)$. As in chapter three, conjugation by elementary matrices is a key technique, so we now give the following lemma.

Lemma 6.1. Let $t_k = t(\Lambda_k,\Phi_k,i_k,f_k)$, $k = 1,2$ be two elementary matrices

(i) if $\Lambda_1 \cap \Phi_2$ and $\Lambda_2 \cap \Phi_1$ are empty then $[t_1,t_2] = 1$,

(ii) if $\Lambda_2 \cap \Phi_1$ is empty but $\Lambda_1 \cap \Phi_2$ is not empty then $[t_1,t_2] = t(\Lambda_2,\Phi_1',j,g)$ where $j = i_1 \circ i_2$, $\Phi_1' = i_1(\Lambda_1 \cap \Phi_2)$ and

$$g(\beta,\lambda) = \sum_{\varphi \in i_2(\lambda) \cap \Lambda_1} f_1(\beta,\varphi) f_2(\varphi,\lambda),$$

(iii) if $\Lambda_2 \cap \Phi_1$ is not empty but $\Lambda_1 \cap \Phi_2$ is empty then $[t_1,t_2] = t(\Lambda_1,\Phi_2',j,g)$ where $j = i_2 \circ i_1$, $\Phi_2' = i_2(\Lambda_2 \cap \Phi_1)$ and

$$g(\beta,\lambda) = -\sum_{\varphi \in i_1(\lambda) \cap \Lambda_2} f_2(\beta,\varphi) f_1(\varphi,\lambda).$$

Proof. We examine the effect of $[t_1,t_2]$ on the canonical basis of M. (i) is obvious.

(ii) Here we assume that $\Lambda_2 \cap \Phi_1$ is empty while $\Lambda_1 \cap \Phi_2$ is not.

$$t_2(e_\omega) = e_\omega + \sum_{\varphi \in i_2(\omega)} e_\varphi f_2(\varphi,\omega) \qquad (i_2(\omega) \subseteq \Phi_2)$$

$$t_1t_2(e_\omega) = e_\omega + \sum_{\varphi \in i_1(\omega)} e_\varphi f_1(\varphi,\omega) + \sum_{\varphi \in i_2(\omega)} e_\varphi f_2(\varphi,\omega)$$
$$+ \sum_{\substack{\varphi \in i_2(\omega) \cap \Lambda_1 \\ \alpha \in i_1(i_2(\omega))}} e_\alpha f_1(\alpha,\varphi) f_2(\varphi,\omega)$$

and so

$$t_1^{-1}t_2^{-1}t_1t_2(e_\omega) = e_\omega + \sum_{\substack{\varphi \in i_2(\omega) \cap \Lambda_1 \\ \alpha \in i_1(i_2(\omega))}} e_\alpha f_1(\alpha,\varphi) f_2(\varphi,\omega)$$

Thus, $[t_1,t_2]$ has the required form since $[t_1,t_2]e_\omega = e_\omega$ for $\omega \notin \Lambda_2$.

(iii) Here we assume that $\Lambda_2 \cap \Phi_1$ is not empty but $\Lambda_1 \cap \Phi_2$ is empty.

$$t_1t_2(e_\omega) = e_\omega + \sum_{\varphi \in i_1(\omega)} e_\varphi f_1(\varphi,\omega) + \sum_{\varphi \in i_2(\omega)} e_\varphi f_2(\varphi,\omega)$$

$$t_2^{-1}t_1t_2(e_\omega) = e_\omega + \sum_{\varphi \in i_1(\omega)} e_\varphi f_1(\varphi,\omega) - \sum_{\substack{\alpha \in i_2(i_1(\omega)) \\ \varphi \in i_1(\omega) \cap \Lambda_2}} e_\alpha f_2(\alpha,\varphi) f_1(\varphi,\omega)$$

Thus $[t_1,t_2]$ has the required form and this completes the proof of the lemma.

This lemma now enables us to state and prove

Corollary 6.1. Let $\underline{a}$, $\underline{b}$ be infinite cardinals with $\underline{a} \leq \underline{b} \leq \underline{w}$. $[E(\Omega,R,\underline{a}),E(\Omega,R,\underline{b})] = E(\Omega,R,\underline{a})$.

Proof. Let $A = [E(\Omega,R,\underline{a}),E(\Omega,R,\underline{b})]$. Let $t_1 = t(\Lambda_1,\Phi_1,i_1,f_1)$ be an elementary matrix in $E(\Omega,R,\underline{a})$ and let $t_2 = t(\Lambda_2,\Phi_2,i_2,f_2) \in E(\Omega,R,\underline{b})$. Then card $\Phi_1 < \underline{a}$ and card $\Phi_2 < \underline{b}$. If either $\Lambda_2 \cap \Phi_1$ is empty or $\Lambda_1 \cap \Phi_2$ is empty then $[t_1,t_2] \in E(\Omega,R,\underline{a})$ since, using the notation of the lemma, $[t_1,t_2] = t(*,\Phi',*,*)$ where $\Phi' = i_1(\Lambda_1 \cap \Phi_2) \subseteq \Phi_1$ or

$\Phi' = i_2(\Lambda_2 \cap \Phi_1)$ respectively. In either case card $\Phi' < \underline{a}$. We now need to consider the case in which neither $\Lambda_1 \cap \Phi_2$ nor $\Lambda_2 \cap \Phi_1$ is empty. Notice that

$$t(\Lambda_1,\Phi,*,*)t(\Lambda_2,\Phi,*,*) = t(\Lambda_1 \cup \Lambda_2,\Phi,*,*)$$

and

$$t(\Lambda,\Phi_1,*,*)t(\Lambda,\Phi_2,*,*) = t(\Lambda,\Phi_1 \cup \Phi_2,*,*).$$

Write Λ_1 as the disjoint union $\Lambda_1' \cup (\Lambda_1 \cap \Phi_2)$ so that

$$[t_1,t_2] = [t_1',t_2]^{t_1''}[t_1'',t_2]$$

where $t_1' = t(\Lambda_1',\Phi_1',i_1,f_1)$ and $t_1'' = t(\Lambda_1 \cap \Phi_2,\Phi_2'',i_1,f_1)$ with $\Phi_1' = i_1(\Lambda_1')$, $\Phi_2'' = i_1(\Lambda_1 \cap \Phi_2)$. We see that $[t_1',t_2]^{t_1''} \in E(\Omega,R,\underline{a})$. Now write $t_2 = t_2't_2''$ where $t_2' = t(\Lambda_2',\Phi_2',i_2,f_2)$ and $t_2'' = t(\Lambda_2 \cap \Phi_1,\Phi_2'',i_2,f_2)$ with $\Phi_2' = i_2(\Lambda_2')$, $\Phi_2'' = i_2(\Lambda_2 \cap \Phi_1)$ and where Λ_2 is the disjoint union $\Lambda_2' \cup (\Lambda_2 \cap \Phi_1)$. Then

$$[t_1'',t_2] = [t_1'',t_2't_2''] = [t_1'',t_2''][t_1'',t_2']^{t_2''}.$$

But $[t_1'',t_2'] \in E(\Omega,R,\underline{a})$ by the lemma and since card $\Phi_2'' < \underline{a}$ it follows that t_2'' and $[t_1'',t_2'']$ also belong to $E(\Omega,R,\underline{a})$. We deduce that $A \in E(\Omega,R,\underline{a})$.

Let $t(\Lambda,\Phi,i,f)$ be an elementary matrix in $E(\Omega,R,\underline{a})$. Suppose that there exists $\Phi' \subset \Omega$ with $\Phi \cap \Phi'$ empty and card $\Phi' =$ card Φ. Let j be a bijection $\Phi' \to \Phi$ and define $g:\Phi' \times \Phi \to R$ by $g(\alpha,\beta) = -1$ for all $(\alpha,\beta) \in \Phi' \times \Phi$. Then we see that

$$t(\Lambda,\Phi,i,f) = [t(\Lambda,\Phi',i',f'),t(\Phi',\Phi,j,g)]$$

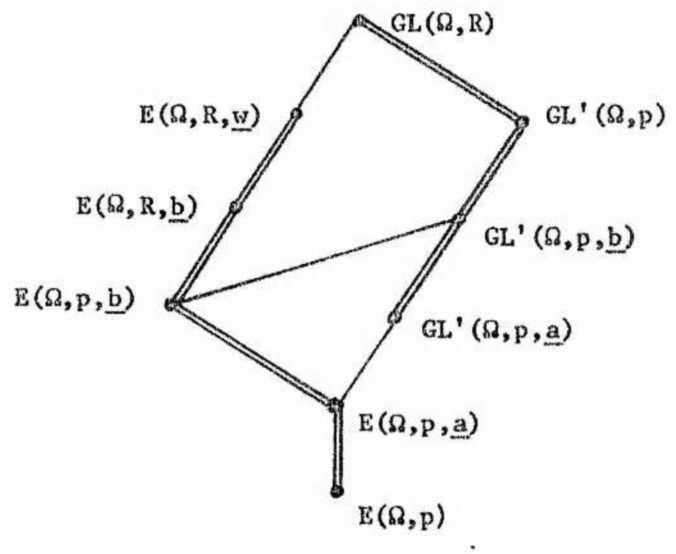
where $i'(\lambda) = j^{-1}(i(\lambda))$, $\lambda \in \Lambda$ and $f'(\varphi',\lambda) = f(\varphi,\lambda)$ where $j(\varphi') = \varphi$. If we cannot find such a set Φ' then write Φ as the disjoint union $\Phi_1 \cup \Phi_2$ with card $\Phi_1 =$ card Φ_2 and proceed in two stages. This completes the proof of the corollary.

Notice that Corollary 6.1 shows that for each infinite $\underline{a} \leq \underline{w}$, $E(\Omega,R,\underline{a})$ is perfect. In fact we are also able to deduce

Corollary 6.2. Let $\underline{a}$, $\underline{b}$ be infinite cardinals with $\underline{a} \leq \underline{b} \leq \underline{w}$ and let p be a two sided ideal of R. $E(\Omega,p,\underline{a}) = [E(\Omega,p,\underline{a}),E(\Omega,R,\underline{b})]$.

Proof. This follows immediately from Corollary 6.1 and Lemma 6.1 since p is a two sided ideal and since $E(\Omega,p,\underline{a})$ is a normal subgroup of $E(\Omega,R,\underline{a})$.

These corollaries show that we have an ascending chain $\{E(\Omega,R,\underline{a}) : \underline{a} \leq \underline{w}\}$ of normal subgroups of $E(\Omega,R,\underline{w})$. The results of chapter three show that when R is d-finite each normal subgroup H of $GL(\Omega,R)$ determines a unique ideal p of R such that H lies between $E(\Omega,p)$ and $GL'(\Omega,p)$. We shall show that H also determines infinite cardinals $\underline{a}$ and $\underline{b}$ such that H lies between $E(\Omega,p,\underline{a})$ and $GL'(\Omega,p,\underline{b})$. The following diagram, which shows the relationships between $E(\Omega,p)$, $E(\Omega,p,\underline{a})$, $GL'(\Omega,p,\underline{b})$ and $GL'(\Omega,p)$, demonstrates that this new classification of the normal subgroups of $GL(\Omega,R)$ is an improvement.



We shall now show how we construct the infinite cardinals $\underline{a}$ and $\underline{b}$ mentioned above. <u>For the rest of this chapter, we shall assume that R is d-finite.</u>

Let $X \in E(\Omega,R,\underline{w})$, $X \neq 1$. Since R is d-finite, $J(X)$ is finitely generated and so there exist entries y_j of $X-I$ such that the two sided ideal generated by $\{y_j\}$ is $J(X)$. We shall call $\{y_j\}$ an elementary generating set for $J(X)$ if the two sided ideal generated by $\{y_j\}$ is $J(X)$ but any proper subset of $\{y_j\}$ does not generate $J(X)$ in this way. Let $A_n(X)$ index the collection of elementary generating sets of $J(X)$ of finite cardinality n and let $C(X)$ be the set of all cardinalities of finite elementary generating sets for $J(X)$. Let $N = \{1,\ldots,n\}$. For every $n \in C(X)$, well order $A_n(X)$ and N by $\leq_1$ and $\leq_2$ respectively. Let Φ, Λ index the non-zero rows and columns of $X-I$ respectively and well order Φ, Λ by $\leq_3$ and $\leq_4$ respectively. Let $n \in C(X)$; for each $\alpha \in A_n(X)$ let $\{y_{i\alpha}\}_1^n$ denote the 'αth' generating set of cardinality n. We shall now construct cardinals $\underline{a}(X,\alpha,\leq_1,\leq_2,\leq_3,\leq_4)$ and $\underline{a}(X)$ by the following algorithm. (We shall abbreviate $\underline{a}(X,\alpha,\leq_1,\leq_2,\leq_3,\leq_4)$ to $\underline{a}(X,\alpha,\leq)$),

let $N' \leftarrow N$

L1: let i be the first member of N'

let $A_n' \leftarrow A_n(X)$

let $\Phi' \leftarrow \Phi$, $\Lambda' \leftarrow \Lambda$

L2: let α be the first member of A_n', $\Lambda(i,\alpha,\leq) \leftarrow \phi$

let $\Phi'' \leftarrow \Phi'$, $A_n' \leftarrow A_n' - \{\alpha\}$

L3: let φ be the first member of Φ''

let $\Lambda'' \leftarrow \Lambda'$, $\Phi'' \leftarrow \Phi'' - \{\varphi\}$

L4: let λ be the first member of Λ''

let $\Lambda'' \leftarrow \Lambda'' - \{\lambda\}$

if $(X-I)_{\varphi\lambda} \neq y_{i\alpha}$ then goto L5

comment: continue row search for elementary generators

$\Lambda(i,\alpha,\underline{<}) \leftarrow \Lambda(i,\alpha,\underline{<}) \cup \{(\varphi,\lambda)\}$

comment: we have found an elementary generator at position (φ,λ)

let $\Lambda' \leftarrow \Lambda' - \{\lambda\}$, $\Phi' \leftarrow \Phi' - \{\varphi' : (X-I)_{\varphi'\lambda} \neq 0\}$

comment: delete this position from $\Lambda' \times \Phi'$

$\Phi'' \leftarrow \Phi'$, goto L6

comment: omit rest of row and search next non-trivial row

L5:if $\Lambda'' \neq \phi$ then goto L4

L6:if $\Phi'' \neq \phi$ then goto L3

comment: start next row search if $\Phi'' \neq \phi$, otherwise, look at next elementary generating set

if $A'_n \neq \phi$, $\Phi' \neq \phi$ and $\Lambda' \neq \phi$ then goto L2

$\underline{a}(X,i,\alpha,\underline{<})$ = card $\Lambda(i,\alpha,\underline{<})$

let $N' \leftarrow N' - \{i\}$

if $N' \neq \phi$ then goto L1, otherwise

for each $\alpha \in A_n(X)$ let $\underline{a}(X,\alpha,\underline{<}) = \min_{i \in N} \underline{a}(X,i,\alpha,\underline{<})$.

The set of cardinals greater than all the sums

$$\sum_{\alpha \in A_n(X)} \underline{a}(X,\alpha,\underline{<})$$

for all $n \in C(X)$ and all orders $\underline{<}$ of $A_n(X)$, N, Φ and Λ has a least member. Denote this cardinal by $\underline{a}(X)$.

Let $H \underline{<} E(\Omega,R,\underline{w})$. Since R is d-finite we note that

$$J(H) = J(X_1) + \ldots + J(X_n)$$

for $X_i \in H$, $X_i \neq 1$, $i = 1,\ldots,n$. We shall call $\{X_1,\ldots,X_n\}$ a set of matrix generators for $J(H)$ if $J(H) = J(X_1) + \ldots + J(X_n)$ but no proper subset of $\{X_1,\ldots,X_n\}$ generates $J(H)$ in this way. Let $A_n(H)$ index the collection of sets of matrix generators for $J(H)$ of finite cardinality n and let $C(H)$ be the set of cardinalities of finite matrix generating sets. Let $n \in C(H)$ and $\alpha \in A_n(H)$. Then $\{X_{1\alpha},\ldots,X_{n\alpha}\}$ is a matrix generating set. Define

$$\underline{a}(\alpha,n,H) = \min_{1 \leq i \leq n} \underline{a}(X_{i\alpha}).$$

The set of cardinals $\underline{a}(\alpha,n,H)$, for all $\alpha \in A_n(H)$ and $n \in C(H)$ has a supremum. If it is infinite denote it by $\underline{a}(H)$; otherwise put $\underline{a}(H) = \aleph_0$.

Let $H \leq GL(\Omega,R)$. For $X \in H$, $X \neq 1$, let $\underline{r}(X)$ be the cardinal of the set indexing the rows of $X-I$ that are non-zero. The set of cardinals greater than $\underline{r}(X)$, for all $X \in H$, $X \neq 1$, has a least member. If this least member is infinite, denote it by $\underline{r}(H)$; otherwise put $\underline{r}(H) = \aleph_0$. We shall call $\underline{r}(H)$ the rank of H and $\underline{r}(X)$ the rank of X. We see that $\underline{r}(E(\Omega,R,\underline{a})) = \underline{a}$ and every $X \in E(\Omega,R,\underline{a})$ has rank less than $\underline{a}$. If $\underline{r}(H) > \underline{w}$ put $GL'(\Omega,J(H),\underline{r}(H)) = GL'(\Omega,J(H))$.

Lemma 6.2. Let $H \leq GL(\Omega,R)$. If $H \leq E(\Omega,R,\underline{w})$ or if H is normalized by $E(\Omega,R)$ then there exist an infinite cardinal $\underline{b}$ and a two sided ideal p of R such that

$$H \leq GL'(\Omega,p,\underline{b}).$$

Moreover, $\underline{r}(H)$ and $J(H)$ are minimal amongst all possible $\underline{b}$ and p, respectively, in this inclusion.

Proof. Let H satisfy the hypotheses of the lemma. From chapter three we see that $H \leq GL'(\Omega,J(H))$. For $X \in H$, $\underline{r}(X) < \underline{r}(H)$ and

$J(X) \leq J(H)$. Thus $X \in GL'(\Omega, J(X), \underline{r}(H)) \leq GL'(\Omega, J(H), \underline{r}(H))$. We conclude that $H \leq GL'(\Omega, J(H), \underline{r}(H))$.

If $H \leq GL'(\Omega, p, \underline{v})$, first notice that $GL'(\Omega, p, \underline{v}) \leq GL'(\Omega, p)$ so that, from chapter three $p \geq J(H)$. Next notice that for all $X \in H$ $\underline{r}(X) < \underline{v}$ and so $\underline{r}(H)$ and $J(H)$ are minimal.

Lemma 6.3. If $\underline{w} > \aleph_0$ then R is a simple ring if and only if for all $H \leq E(\Omega, R, \underline{w})$, $\underline{a}(H) = \underline{r}(H)$.

Proof. Using the notation of the construction of $\underline{a}(X)$, we see that

$$\underline{r}(X) \geq \sum_{\alpha \in A_n(X)} \underline{a}(X, \alpha, \leq)$$

for all $n \in C(X)$ and all orders $\leq$ of $A_n(X)$, N, Φ and Λ. Since $\underline{r}(H) > \underline{r}(X)$ we see that $\underline{r}(H) \geq \underline{a}(X)$, for all $X \in H$ and hence $\underline{r}(H) \geq \underline{a}(H)$.

Suppose that R is simple. We need only show that $\underline{r}(H) \leq \underline{a}(H)$. Let $X \in H$, $X \neq 1$ and suppose that $\underline{r}(X)$ is infinite. Since R is simple any non-zero entry of X-I will generate R. Let Λ index the non-zero columns of X-I and Φ index the non-zero rows of X-I. Well order Λ and Φ. Let φ be the first element of Φ and λ be the first element of Λ. Then $\{(X-I)_{\varphi\lambda}\}$ will be an elementary generating set. Delete $\{\varphi' : (X-I)_{\varphi'\lambda} \neq 0\}$ and λ from Φ and Λ respectively. CardΦ is still equal to $\underline{r}(X)$ since X is column finite. We can repeat this process and construct our set $\Lambda(X, \alpha, \leq)$ of cardinality $\underline{r}(X)$. Hence $\underline{a}(X) = \underline{r}(X)$ since our choice of well ordering was arbitrary. It follows that $\underline{a}(H) = \underline{r}(H)$. (If $\underline{r}(X)$ is finite for all $X \in H$, $X \neq 1$, then $\underline{a}(H) = \underline{r}(H) = \aleph_0$.)

Now suppose that $\underline{a}(H) = \underline{r}(H)$, for all $H \leq E(\Omega, R, \underline{w})$ and suppose that R is not simple. There exist non-zero $x, y \in R$ such that

$(x) < (x,y)$. ((x) denotes the two sided ideal generated by x.) Let $t = t(\Lambda,\Phi,i,f)$ be the elementary matrix in which Φ is countable, $i:\Lambda \to \Phi$ is a bijection and $f(\varphi,\lambda) = x$, for only finitely many λ and $f(\varphi,\lambda) = y$ elsewhere. Let H be the subgroup of $E(\Omega,R,\underline{w})$ generated by t. Then $J(H) = R$, $\underline{a}(H) = \aleph_0$ while $\underline{r}(H) > \aleph_0$, contrary to hypothesis. We conclude that R is simple.

Definition 6.7. Let $t = t(\Lambda,\Phi,i,f)$ be an elementary matrix. We shall say that t is $\underline{a}$-sparse if and only if, in the elementary correspondence (Λ,Φ,i), i is a bijection and card $\Phi = \underline{a}$.

We see that $\underline{a}$-sparse elementary matrices have precisely $\underline{a}$ non-trivial rows, precisely $\underline{a}$ non-trivial columns and each row and column has at most one non-trivial entry. The proof of our main theorem hinges on showing that any subgroup of $E(\Omega,R,\underline{w})$ normalized by $E(\Omega,R,\underline{r}(H))$ contains $\underline{a}$-sparse elementary matrices, for some cardinals $\underline{a}$.

Lemma 6.4. Let H be a subgroup of $E(\Omega,R,\underline{w})$ normalized by $E(\Omega,R,\underline{r}(H))$. Let $X \in H$, $X \neq 1$, let $n \in C(X)$ and let $\leq$ be some orders on N, $A_n(X)$, Φ and Λ – we use the notation of the construction of $\underline{a}(X)$. Let $1 \leq k \leq n$. For any disjoint equipotent sets Λ' and Φ' of cardinality

$$\underline{b} = \sum_{\alpha \in A_n(X)} \underline{a}(X,\alpha,\leq)$$

partition Λ', Φ' into subsets $\Lambda'(\alpha)$, $\Phi'(\alpha)$ of cardinality $\underline{a}(X,\alpha,\leq)$. Let $i:\Phi' \to \Lambda'$ be a bijection such that for all $\alpha \in A_n(X)$ $i(\Phi'(\alpha)) = \Lambda'(\alpha)$, let $p_\alpha:\Phi'(\alpha) \to \Lambda(k,\alpha,\leq)$ be an injection. Let $\varphi \in \Phi'$ and let $\alpha \in A_n(X)$ be such that $\varphi \in \Phi'(\alpha)$. Define $f:\Phi' \times \Lambda' \to R$ by

$$f(\varphi,\lambda) = \begin{cases} (X-I)_{p_\alpha(\varphi)} & \text{if } \lambda = i(\varphi) \\ 0 & \text{otherwise.} \end{cases}$$

H contains the $\underline{b}$-sparse elementary matrix $t(\Lambda',\Phi',i^{-1}f)$.

Remark. This lemma shows that, for any $X \in H$, H contains the $\underline{b}$-sparse elementary matrix, every non-trivial entry of which is the kth member of some generating set of cardinality n for $J(X)$. Moreover the kth member of any generating set of cardinality n for $J(X)$ is a non-trivial entry of that elementary matrix.

Proof. We shall assume that $\underline{b}$ is infinite, for if not the result follows from the main theorem of chapter three. We first assert that X has at least $\underline{a} = \underline{r}(X)$ non-trivial columns. (Notice that $\underline{a}$ is infinite since $\underline{b}$ is infinite.) This follows from noting that X has $\underline{r}(X)$ non-trivial rows and that X is column finite. We next assert that we can assume that X has at least $\underline{a}$ trivial columns. For suppose not. Define the subset Λ_0 of Ω by

$$\Lambda_0 = \{\lambda \in \Omega : \lambda\text{th row of } X^{-1} \text{ is trivial but } \lambda\text{th column is not}\}.$$

Then card $\Lambda_0 > \underline{a}$ by our supposition. Let Φ_0 be the set that indexes the columns recorded by the elements of $\bigcup_{\alpha \in A_n(X)} \Lambda(k,\alpha,\underline{<})$. Since Φ_0 and Λ_0 are infinite we can assume that there exists a subset Ψ of Λ_0 of cardinality $\underline{a}$ disjoint from Φ_0. If we put $\Lambda_0' = \Lambda_0 - (\Psi \cup \Phi_0)$ then we can also assume that card $\Lambda_0' > \underline{a}$. There exists a surjection $f:\Lambda_0' \to \Phi_0$. Define $t = t(\Lambda_0',\Phi_0,f,1)$. Then

$$tX^{-1}(e_\mu) = X^{-1}(e_\mu) + e_{f(\mu)}, \quad \mu \in \Lambda_0'$$

and so

$$XtX^{-1}(e_\mu) = e_\mu + X(e_{f(\mu)}), \qquad \mu \in \Lambda'_0.$$

Thus

$$[t,X^{-1}]e_\mu = e_\mu + (X-I)e_{f(\mu)}, \qquad \mu \in \Lambda'_0.$$

Moreover, if $\mu \in \Psi$ then $[t,X^{-1}]e_\mu = e_\mu$. Thus $[t,X^{-1}]$ has at least $\underline{a}$ trivial columns and the Λ'_0 columns of $[t,X^{-1}]$ contain the non-trivial entries of the Φ_0 columns of X. In this way we see that the elementary generator $y_{k\alpha}$ can be found in the Λ'_0 columns of $[t,X^{-1}]$. Finally, we notice that $[t,X^{-1}] \in H$ since $t \in E(\Omega,R,\underline{r}(H))$.

Let Λ_0 index the rows recorded by $\bigcup_\alpha \Lambda(k,\alpha,\leq)$; then Λ_0 and Φ_0 are equipotent with card $\Phi_0 \leq \underline{a}$. From the set indexing the trivial columns of X pick equipotent disjoint sets Ψ_1, Ψ_2 with card Ψ_1 = card Φ_0 with Ψ_1 disjoint from Λ_0. Define Φ_1 to be the set

$$\Phi_1 = \{\varphi \in \Omega : X(e_\varphi) \neq e_\varphi \text{ and } (\exists \lambda \in \Lambda_0) : (X-I)_{\lambda\varphi} \neq 0\}.$$

Then $\Phi_0 \subseteq \Phi_1$. Moreover, if $(\lambda,\varphi) \in \bigcup_\alpha \Lambda(k,\alpha,\leq)$ then there is no other $\lambda' \in \Lambda_0$ such that $(X-I)_{\lambda'\varphi} \neq 0$, by the construction of $\Lambda(k,\alpha,\leq)$. There is a bijection $j:\Lambda_0 \to \Psi_1$. Put $t = t(\Lambda_0,\Psi_1,j,1)$. We see that $[t,X] = t(\Phi_1,\Psi_1,k,g)$, for some $(\Phi_1,\Psi_1,k) \in \mathcal{E}$. Moreover, $k|_{\Phi_0}$ is a bijection $\Phi_0 \to \Psi_1$ such that $g(k(\varphi),\varphi)$ is an elementary generator $y_{k\alpha}$. Now let $k':\Psi_2 \to \Phi_0$ be a bijection. We see that

$$[t(\Psi_2,\Phi_0,k',-1),t(\Phi_1,\Psi_1,k,g)] = t(\Psi_2,\Psi_1,k'',g')$$

is card Φ_0 sparse with its non-trivial entries the elementary generators $y_{k\alpha}$. Since card $\Phi_0 \geq \underline{b}$, we see that further conjugations show that H contains the required elementary matrix.

We now use this lemma to establish

Proposition 6.1. Let H be a subgroup of $E(\Omega,R,\underline{w})$ normalized by $E(\Omega,R,\underline{r}(H))$. If $X \in H$, $X \neq 1$ then H contains $E(\Omega,J(X),\underline{a})$, where $\underline{a} = \max(\aleph_0,\underline{a}(X))$.

Proof. In view of the main theorem of chapter three we assume that $\underline{a}(X) > \aleph_0$. Let $C(X)$, N, $A_n(X)$, Λ and Φ have their usual meanings and let $\leq$ be orderings of N, $A_n(X)$, Λ and Φ. Let

$$\underline{b} = \sum_{\alpha \in A_n(X)} \underline{a}(X,\alpha,\leq)$$

and let Λ'', Φ'' be disjoint equipotent subsets of Ω of cardinality $\underline{b}$ with $j:\Lambda'' \to \Phi''$ a bijection. Partition Λ'' and Φ'' into subsets $\Lambda''(\alpha)$ and $\Phi''(\alpha)$ of cardinality $\underline{a}(X,\alpha,\leq)$, for each $\alpha \in A_n(X)$ in such a way that $j(\Lambda''(\alpha)) = \Phi''(\alpha)$. Let i, Λ', Φ', $\Lambda'(\alpha)$ and $\Phi'(\alpha)$ be as in the statement of Lemma 6.4. (We can assume that Λ', Φ' are disjoint from Λ'', Φ''.)

The two sided ideal generated by the αth generating set $\{y_{i\alpha}\}_1^n$ is $J(X)$. Let

$$g:\Phi'' \times \Lambda'' \to J(X)$$

be an elementary mapping. Form the restricted mapping

$$g_\alpha:\Phi''(\alpha) \times \Lambda''(\alpha) \to J(X)$$

by setting $g_\alpha(\varphi,\lambda) = g(\varphi,\lambda)$ where $\lambda \in \Lambda''(\alpha)$. We first show that H contains the $\underline{b}$-sparse elementary matrix $t(\Lambda'',\Phi'',j,g)$.

Since j is a bijection, $g(\varphi,\lambda)$ depends only upon λ. Put

$$g(\varphi,\lambda) = g_\alpha(\varphi,\lambda) = x_{\alpha\lambda}.$$

Then there exist $r_{\alpha k\lambda}$, $s_{\alpha k\lambda} \in R$, $k = 1,\ldots,n$ such that

$$x_{\alpha\lambda} = \sum_{k=1}^{n} r_{\alpha k\lambda} y_{k\alpha} s_{\alpha k\lambda} = \sum_{k=1}^{n} x'_{\alpha k\lambda}, \text{ say.}$$

There exist bijections $i_1:\Lambda'' \to \Lambda'$, $i_2:\Phi' \to \Phi''$ such that $j = i_2 \circ i^{-1} \circ i_1$. Define the elementary mappings $s_\alpha:\Lambda'(\alpha) \times \Lambda''(\alpha) \to R$ and $r_\alpha:\Phi''(\alpha) \times \Phi'(\alpha) \to R$ by

$$s_\alpha(\lambda',\lambda) = \begin{cases} s_{\alpha k\lambda} & \text{if } \lambda' = i_1(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

and

$$r_\alpha(\varphi,\varphi') = \begin{cases} r_{\alpha k\lambda} & \text{if } j(\lambda) = \varphi = i_2(\varphi') \\ 0 & \text{otherwise} \end{cases}$$

and extend these mappings to $s:\Lambda' \times \Lambda'' \to R$, $r:\Phi'' \times \Phi' \to R$. Let $t_1 = t(\Lambda'',\Lambda',i_1,-s)$, $t_2 = t(\Lambda',\Phi',i^{-1},f)$ and $t_3 = t(\Phi',\Phi'',i_2,r)$. Then

$$t(\Lambda'',\Phi'',j,h) = [t_3,[t_1,t_2]] \in H$$

by the lemma. h is the elementary mapping such that $h(\varphi,\lambda) = x'_{\alpha k\lambda}$ where $\lambda \in \Lambda''(\alpha)$. Denote $t(\Lambda'',\Phi'',j,h)$ by t_k. Repeating this argument for $k = 1,\ldots,n$ we see that H contains Πt_k which is just the $\underline{b}$-sparse elementary matrix $t(\Lambda'',\Phi'',j,g)$. Similar conjugations show that H contains every $\underline{c}$-sparse elementary matrix written in $J(X)$, for any $\underline{c} \leq \underline{b}$.

Since R is d-finite we see that there exist $x_i \in J(X)$, $i = 1,\ldots,r$ such that $J(X) = x_1R + \ldots + x_rR$. Thus if $t(\Lambda,\Phi,i,g)$ is an elementary matrix with card $\Phi = \underline{c} < \underline{a}$ we see that

$$g(\varphi,\lambda) = \sum_{i=1}^{r} x_i f_i(\varphi,\lambda)$$

for some $f_i(\varphi,\lambda) \in R$. Pick $\Phi' \subset \Omega$, card $\Phi' = \underline{c}$ and Φ' disjoint from Λ and Φ. Let $k:\Phi \to \Phi'$ be a bijection and (Λ,Φ',j) be an elementary correspondence such that $i(\lambda) = k(j(\lambda))$. Put $t_i = t(\Lambda,\Phi',j,f_i')$ where $f_i'(\varphi',\lambda) = f_i(\varphi,\lambda)$ for $k(\varphi) = \varphi'$. Then if we put

$$t_i' = [t_i, t(\Phi', \Phi k^{-1}, -x_\alpha)]$$

we see that $t_i' \in H$ by our remarks above. However

$$t(\Lambda, \Phi, i, g) = \prod_{i=1}^{r} t_i'$$

and so H contains $t(\Lambda, \Phi, i, g)$.

To complete the proof of the proposition notice that if $\underline{c} < \underline{a}(X)$ then $\underline{c} \leq \sum_{\alpha \in A_n(X)} \underline{a}(X, \alpha, \underline{\leq})$ for some n and some orders $\underline{\leq}$.

Theorem 6.1. Let R be a d-finite ring. If H is a normal subgroup of $E(\Omega, R, \underline{w})$ then there exist infinite cardinals $\underline{a}$, $\underline{b}$ and a unique two sided ideal p of R such that

$$E(\Omega, p, \underline{a}) \leq H \leq GL'(\Omega, p, \underline{b}). \qquad (*)$$

Moreover, $\underline{a}(H)$ is maximal amongst all possible cardinals $\underline{a}$ in (*) and $\underline{r}(H)$ is minimal amongst all possible cardinals $\underline{b}$ in (*).

Proof. Let $t(\Lambda, \Phi, i, f)$ be an elementary matrix written in $J(X)$ and of rank $\underline{b} < \underline{a}(H)$. Since $\underline{b} < \underline{a}(H)$ we see that $\underline{b} < \underline{a}(\alpha, n, H)$ for some $\alpha \in A_n(H)$ and some $n \in C(H)$. However, R is d-finite so $J(H) = J(X_{1\alpha}) + \ldots + J(X_{n\alpha})$ for some $X_{i\alpha} \in H$ and by the construction of $\underline{a}(\alpha, n, H)$, $\underline{b} < \underline{a}(X_{i\alpha})$, $i = 1, \ldots, n$. However $t(\Lambda, \Phi, i, f) = t_1 \ldots t_r$ where each t_i is written in $J(X_{i\alpha})$, $i = 1, \ldots, r$ and $\underline{r}(t_i) \leq \underline{b}$. The proposition shows that H contains $E(\Omega, J(X_{i\alpha}), \underline{a}_i)$ where $\underline{a}_i = \max(\aleph_0, \underline{a}(X_{i\alpha}))$ so it follows that H contains $E(\Omega, J(H), \underline{a}(H))$ and this establishes the first inclusion with $\underline{a} = \underline{a}(H)$ and $p = J(H)$. The second inclusion is immediate from Lemma 6.2.

The fact that $J(H)$ is maximal with respect to the first inclusion is immediate from the definition. From Lemma 6.2, $J(H)$ is minimal with respect to the second inclusion. Thus if $\underline{a}$, $\underline{b}$ and p are such that

$$E(\Omega,p,\underline{a}) \leq H \leq GL'(\Omega,p,\underline{a})$$

we see that $p \leq J(H) \leq p$, that is the ideal $p = J(H)$ is unique. Now suppose that $\underline{a}$ is an infinite cardinal such that

$$E(\Omega,J(H),\underline{a}) \leq H \leq GL'(\Omega,J(H),\underline{r}(H))$$

and also suppose that $\underline{a} > \underline{a}(H)$. Since R is d-finite $J(H) = x_1R + \ldots + x_rR$ for some $x_i \in J(H)$, $i = 1,\ldots,r$. Define $t_j = t(\Lambda,\Phi,i,f_j)$ to be the $\underline{a}(H)$-sparse elementary matrix all of whose non-trivial entries are x_j. Put $t = t_1\ldots t_r$. Then $\underline{a}(t) = \underline{a}(H)$ and $J(t) = J(H)$. Moreover, since $\underline{a} > \underline{a}(H) = \underline{a}(t)$ we see that $t \in H$, contrary to the inequality $\underline{a}(H) > \underline{a}(X)$, for all $X \in H$. We conclude that $\underline{a} \leq \underline{a}(H)$ and this completes the proof of the theorem.

Corollary 6.3. If R is a simple ring then the only proper normal subgroups of $E(\Omega,R,\underline{w})$ are the groups $E(\Omega,R,\underline{a})$, for all infinite cardinals $\underline{a} < \underline{w}$.

Proof. Let $\underline{w} = \aleph_\nu$, for some ordinal ν. We proceed by induction on ν. If $\nu = 0$ then the result follows from chapter three. Thus take as inductive hypothesis that for all $\beta < \alpha$ the only proper normal subgroups of $E(\Omega,R,\aleph_\beta)$ are the groups $E(\Omega,R,\aleph_\gamma)$ for all $\gamma < \beta$. We shall now abbreviate $E(\Omega,R,\aleph_\gamma)$ to $E(\gamma)$. First suppose that α is not a limit ordinal. It follows from Proposition 6.1 that $E(\alpha)/E(\alpha-1)$ is simple. Hence, if H is a proper normal subgroup of $E(\alpha)$, $H \cap E(\alpha-1) = H$ and so H is an $E(\gamma)$, for some $\gamma < \alpha$, by our inductive hypothesis. Now suppose that α is a limit ordinal. Then

$$E(\alpha) = \bigcup_{\beta < \alpha} E(\beta).$$

Let H be a proper normal subgroup of $E(\alpha)$. We assertthat there exist $\gamma, \beta < \alpha$ with $\gamma < \beta$ such that $H \cap E(\varsigma) = E(\gamma)$ for all $\beta \leq \varsigma < \alpha$. For if $H \cap E(\beta) = E(\beta)$ for all $\beta < \alpha$ then

$$H = H \cap \bigcup_{\beta < \alpha} E(\beta) = \bigcup_{\beta < \alpha} (H \cap E(\beta)) = \bigcup_{\beta < \alpha} E(\beta) = E(\alpha)$$

contrary to the choice of H. Thus for some $\beta < \alpha$, $H \cap E(\beta) = E(\gamma)$, for some $\gamma < \beta$. Now let $\varsigma > \beta$, $\varsigma < \alpha$. If $H \cap E(\varsigma) = E(\xi)$ for some $\xi > \gamma$ then

$$H \cap E(\beta) = H \cap E(\varsigma) \cap E(\beta) = E(\xi) \cap E(\beta) > E(\gamma).$$

Thus

$$H = \bigcup_{\beta < \alpha} (H \cap E(\beta)) = E(\gamma).$$

We conclude that for all $\alpha \leq \nu$ the only normal subgroups of $E(\alpha)$ are the groups $E(\beta)$ for $0 \leq \beta \leq \alpha$. This completes the proof of the corollary.

Corollary 6.4. Let R be a d-finite ring. If H is a subgroup of $GL(\Omega, R)$ that is normalized by $E(\Omega, R, \underline{w})$ then there exist infinite cardinals $\underline{a}$, $\underline{b}$ and a unique two sided ideal p of R such that

$$E(\Omega, p, \underline{a}) \leq H \leq GL'(\Omega, p, \underline{b}). \qquad (*)$$

Moreover, $\underline{a}(H \cap E(\Omega, R, \underline{w}))$ is maximal amongst all the possible cardinals $\underline{a}$ in (*) and $\underline{r}(H)$ is minimal amongst all possible cardinals $\underline{b}$ in (*).

Proof. Let H be as in the statement of the corollary and suppose that H is central. Then $1 = E(\Omega, 0, \underline{w}) \leq H \leq GL'(\Omega, 0)$. Now suppose that H is not central. Then $H_0 = H \cap E(\Omega, R, \underline{w}) \neq 1$. From the theorem we know that $E(\Omega, J(H_0), \underline{a}(H_0)) \leq H_0$ and from Lemma 6.2 we know that $H \leq GL'(\Omega, J(H), \underline{r}(H))$. Since $J(H_0) = J(H)$ we see that

$$E(\Omega, J(H), \underline{a}(H_0)) \leq H \leq GL'(\Omega, J(H), \underline{r}(H)).$$

The minimality of $\underline{r}(H)$ is immediate from Lemma 6.2. The maximality of $\underline{a}(H_0)$ follows from noting that if $E(\Omega, J(H), \underline{c}) \leq H$ for some $\underline{c} > \underline{a}(H_0)$, $\underline{c} \leq \underline{w}$, then $E(\Omega, J(H_0), \underline{c}) \leq H_0$. But $\underline{c} \leq \underline{a}(H_0)$ from the theorem and this contradiction completes the proof of the corollary.

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